k-FIBONACCI NUMBERS AND k-LUCAS NUMBERS IN BEATTY SEQUENCES GENERATED BY POWERS OF METALLIC MEANS

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ABSTRACT. For each positive integer k, denote the metallic mean $\left(k+\sqrt{k^2+4}\right)/2$ by α_k. In this article, we give some new identities involving the k-Fibonacci numbers, the k-Lucas numbers, metallic means, the floor function, and fractional parts. We also provide some properties of the Beatty sequence $B\left(\alpha_k^n\right)$ generated by α_k^n, where n is any positive integer. Then these properties are used to show connections between k-Fibonacci and k-Lucas numbers and the sequence $B\left(\alpha_k^n\right)$.

1. Introduction

For each $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the largest integer not exceeding x, $\{x\} = x - \lfloor x \rfloor$ the fractional part of x, and $B(x) := (\lfloor bx \rfloor)_{b \geq 1}$ the Beatty sequence generated by x. Depending on the context, we also write $B(x)$ to denote the Beatty set $\{\lfloor bx \rfloor \mid b \in \mathbb{N}\}$. A famous theorem, called Beatty's theorem [2, 3], states that if x, y are positive irrational numbers and $1/x + 1/y = 1$, then $B(x)$ and $B(y)$ form a partition of $\mathbb{N}$, that is, $B(x) \cup B(y) = \mathbb{N}$ and $B(x) \cap B(y) = \emptyset$.

Two well-known Beatty sequences $B(\phi)$ and $B(\phi^2)$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, are called lower and upper Wythoff sequences, respectively, and their combinatorial properties have been extensively studied; see for example in [1, 6, 7, 12]. However, there are only a few arithmetic results concerning sumsets associated with $B(\phi)$ and $B(\phi^2)$. Pongsriiam and his coauthors [10, 14, 15] have recently started the investigation on sumsets associated with Beatty sequences. To do so, they studied relations between Fibonacci numbers and the sets $B(\phi)$ and $B(\phi^2)$ and used these relations to obtain their main results. Dekking [4] and Shallit [16, 17] also extended some of the results in their articles [10, 14, 15].

Our purpose is to extend the results on the Fibonacci numbers, $B(\phi)$, and $B(\phi^2)$ to the case of Lucas sequences $(U_n(a,b))_{n\geq 1}$ and $(V_n(a,b))_{n\geq 1}$ of the first and second kinds, respectively. Recall that these sequences are defined by the recurrence relations

$$U_0 = 0,\ U_1 = 1,\ U_n = aU_{n-1} + bU_{n-2} \text{ for } n \geq 2,$$
$$V_0 = 2,\ V_1 = a,\ V_n = aV_{n-1} + bV_{n-2} \text{ for } n \geq 2,$$

where a and b are arbitrary but fixed relatively prime integers.

We find that it is possible to extend many identities for the Fibonacci numbers to the case of U_n and V_n, and then obtain some connections between U_n, V_n, and $B(\alpha)$, where α is a certain root of the characteristic polynomial $x^2 - ax - b$ of $(U_n(a,b))_{n\geq 1}$ and $(V_n(a,b))_{n\geq 1}$. The positive quadratic irrational numbers that are the positive solutions of quadratic equations of the form $x^2 - ax - b = 0$ are called metallic means [18, 19]. Nevertheless, this possibility occurs when $b = 1$, whereas the general case seems complicated. So, we focus only on the simpler case where $a = k$ is any positive integer and $b = 1$, and we postpone the general case for future research. Therefore, we let $F_{k,n} = U_n(k,1)$ and $L_{k,n} = V_n(k,1)$ for all $n \geq 0$, and let α_k and β_k be positive and negative roots of the characteristic polynomial $x^2 - kx - 1$ of the sequences $(F_{k,n})_{n\geq 1}$ and $(L_{k,n})_{n\geq 1}$. We call $F_{k,n}$, $L_{k,n}$, and α_k the k-Fibonacci numbers, k-Lucas numbers, and a metallic mean, respectively. In particular, the sequences $(F_{1,n})_{n\geq 1}$, $(F_{2,n})_{n\geq 1}$, $(F_{3,n})_{n\geq 1}$, $(L_{1,n})_{n\geq 1}$, $(L_{2,n})_{n\geq 1}$, and $(L_{3,n})_{n\geq 1}$ are the sequences A000045, A000129, A006190, A000032, A002203, and A006497, respectively, in the On-Line Encyclopedia of Integer Sequences (OEIS) [13]. We give various identities for $F_{k,n}$ and $L_{k,n}$, show some properties of $B(\alpha_k^m)$, where m is any positive integer, and then we provide some connections between $F_{k,n}$, $L_{k,n}$, and $B(\alpha_k^m)$.

2. Notations and Preliminary Results

Before proceeding further, let us recall the well-known Binet formula, which holds in the general case of Lucas sequences of the first and second kinds. Therefore, it also holds for $F_{k,n}$ and $L_{k,n}$. That is

$$F_{k,n} = \frac{\alpha_k^n - \beta_k^n}{\alpha_k - \beta_k} \quad \text{and} \quad L_{k,n} = \alpha_k^n + \beta_k^n \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

where $\alpha_k = \left(k + \sqrt{k^2+4}\right)/2$ and $\beta_k = \left(k - \sqrt{k^2+4}\right)/2$ are the roots of the characteristic polynomial $x^2 - kx - 1 = 0$. Because k is a fixed positive integer, we sometimes write $\alpha = \alpha_k$, $\beta = \beta_k$, $F_{k,n} = F_n$, and $L_{k,n} = L_n$ if no confusion arises. Let us also provide some remarks on properties of α and β. Note that, because k is positive, we obtain $-1 < \beta < 0 < \alpha$ and $|\beta| < \alpha$. Thus, if $m \leq n$ are positive even integers, then $0 < \beta^n \leq \beta^m$, and if $m \leq n$ are positive odd integers, then $\beta^m \leq \beta^n < 0$. Moreover, one can check that $\alpha = \alpha_k$ and $\beta = \beta_k$ are increasing functions of k. So $-0.5 < \beta$ for all $k \geq 2$. Furthermore, $\alpha^2 = k\alpha + 1$, $\alpha^3 = k^2\alpha + k + \alpha$, $\beta^2 = k\beta + 1$, $\alpha = -\frac{1}{\beta}$, $\alpha + \beta = k$, $\alpha\beta = -1$, and $\alpha - \beta = \sqrt{k^2+4}$. These results will be applied throughout this paper sometimes without further reference.

Lemma 2.1. *Let n be an integer and let x and y be real numbers. Then, the following statements hold.*

(i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$.
(ii) $\{n + x\} = \{x\}$.
(iii) $0 \leq \{x\} < 1$.
(iv) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if} \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if} \{x\} + \{y\} \geq 1. \end{cases}$
(v) $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ *if x is not an integer.*
(vi) $\{-x\} = 1 - \{x\}$ *if x is not an integer.*
(vii) $\{\{x + y\}\} = \{\{x\} + \{y\}\}$.

Proof. The results in (i) to (vi) are well known, and their details can be found in [8, Chapter 3]. For (vii), recall the identity in [10, Lemma 2.4] that

$$\{x_1 + x_2 + \cdots + x_n\} = \{\{x_1\} + \{x_2\} + \cdots + \{x_n\}\}.$$

Therefore, $\{\{x + y\}\} = \{\{x\} + \{y\}\}$. □

Lemma 2.2. *Let n be a positive integer. Then, the following statements hold.*

(i) $L_{k,n} = F_{k,n+1} + F_{k,n-1}$.
(ii) $\alpha_k^n = \alpha_k F_{k,n} + F_{k,n-1}$ *and* $\beta_k^n = \beta_k F_{k,n} + F_{k,n-1}$.
(iii) $\sqrt{k^2+4}\alpha_k^n = \alpha_k L_{k,n} + L_{k,n-1}$ *and* $-\sqrt{k^2+4}\beta_k^n = \beta_k L_{k,n} + L_{k,n-1}$.
(iv) $F_{k,n} = \alpha_k F_{k,n-1} + \beta_k^{n-1}$.
(v) $L_{k,n} = \alpha_k L_{k,n-1} - \sqrt{k^2+4}\beta_k^{n-1}$.

Proof. For convenience, we write F_n, L_n, α, and β instead of $F_{k,n}$, $L_{k,n}$, α_k, and β_k, respectively. By Binet's formula and noting that $\alpha\beta = -1$, we see that the right side of (i) is

$$\frac{\alpha^n\left(\alpha+\frac{1}{\alpha}\right)-\beta^n\left(\beta+\frac{1}{\beta}\right)}{\alpha-\beta}=\frac{\alpha^n\left(\alpha-\beta\right)+\beta^n\left(\alpha-\beta\right)}{\alpha-\beta}=L_n.$$

For (ii), we obtain

$$\begin{aligned}\alpha F_n+F_{n-1}&=\alpha\left(kF_{n-1}+F_{n-2}\right)+F_{n-1}\\&=\left(k\alpha+1\right)F_{n-1}+\alpha F_{n-2}\\&=\alpha^2F_{n-1}+\alpha F_{n-2}\\&=\alpha^2\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)+\alpha\left(\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha-\beta}\right)\\&=\frac{\alpha^{n+1}+\alpha^{n-1}}{\alpha-\beta}\\&=\frac{\alpha^n\left(\alpha+\frac{1}{\alpha}\right)}{\alpha-\beta}\\&=\alpha^n.\end{aligned}$$

Then, $\alpha^n - (\beta F_n + F_{n-1}) = (\alpha F_n + F_{n-1}) - (\beta F_n + F_{n-1}) = (\alpha - \beta) F_n = \alpha^n - \beta^n$, which implies $\beta^n = \beta F_n + F_{n-1}$.

For (iii), we see that $\alpha L_n + L_{n-1}$ is equal to

$$\alpha\left(\alpha^n+\beta^n\right)+\left(\alpha^{n-1}+\beta^{n-1}\right)=\alpha^{n+1}-\beta^{n-1}+\alpha^{n-1}+\beta^{n-1}=\alpha^n\left(\alpha-\beta\right)=\sqrt{k^2+4}\alpha^n.$$

Similarly, $\beta L_{k,n} + L_{k,n-1}$ is equal to

$$\beta\left(\alpha^n+\beta^n\right)+\left(\alpha^{n-1}+\beta^{n-1}\right)=-\alpha^{n-1}+\beta^{n+1}+\alpha^{n-1}+\beta^{n-1}=-\beta^n\left(\alpha-\beta\right)=-\sqrt{k^2+4}\beta^n.$$

Multiplying each side of the second identity in (ii) and (iii) by α, we obtain (iv) and (v), respectively. □

Lemma 2.3. *For positive integers m and n with $m \geq n$, the following equalities hold.*

(i) $F_{k,m+n} = F_{k,m-1}F_{k,n} + F_{k,m}F_{k,n+1}$.
(ii) $L_{k,n+m} = F_{k,m-1}L_{k,n} + F_{k,m}L_{k,n+1}$.

Proof. These identities can be proved by applying Binet's formula to the right side of each equation, and then doing a straightforward algebraic manipulation. The details are left to the reader. □

To simplify statements and notations that will appear from now on, we will use the Iverson notation $[P]$, defined by

$$[P]=\begin{cases}1, & \text{if } P \text{ holds;}\\0, & \text{otherwise,}\end{cases}$$

where P is a mathematical statement.

Lemma 2.4. *For each positive integer n,*

$$L_{k,n} = \lfloor \alpha_k^n \rfloor + [n \equiv 0 \pmod 2].$$

Proof. Let n be a positive integer. We denote $L_{k,n}$, α_k, and β_k by L_n, α, and β, respectively. If n is even, then

$$L_n - 1 = \alpha^n + \beta^n - 1 < \alpha^n < \alpha^n + \beta^n = L_n$$

because $0 < \beta^n < 1$. On the other hand, if n is odd, then

$$L_n = \alpha^n + \beta^n < \alpha^n < \alpha^n + \beta^n + 1 = L_n + 1$$

because $-1 < \beta^n < 0$. Therefore, $L_n = \lfloor \alpha^n \rfloor + [n \equiv 0 \pmod 2]$. □

Corollary 2.5. *For each positive integer n,*

$$\lfloor L_{k,n}\alpha_k^n \rfloor = L_{k,2n} - 2[n \equiv 1 \pmod 2].$$

Proof. Let n be a positive integer. Again, we denote $L_{k,n}$, α_k, and β_k by L_n, α, and β, respectively. By Lemma 2.4, we immediately obtain that

$$\lfloor L_n\alpha^n \rfloor = \lfloor (\alpha^n + \beta^n)\,\alpha^n \rfloor = \left\lfloor \alpha^{2n} + (-1)^n \right\rfloor = \left\lfloor \alpha^{2n} \right\rfloor + (-1)^n = L_{2n} - 1 + (-1)^n.$$

Thus, $\lfloor L_n\alpha^n \rfloor = L_{2n} - 2[n \equiv 1 \pmod 2]$ as required. □

Lemma 2.6. *Let m and n be positive integers. Then, the following statements are valid.*

(i) *If $n \geq m$, then $\lfloor F_{k,n}\alpha_k^m \rfloor = F_{k,n+m} - [n \equiv 0 \pmod 2]$.*
(ii) *If $n \geq m+1$, then $\lfloor L_{k,n}\alpha_k^m \rfloor = L_{k,n+m} - [n \equiv 1 \pmod 2]$.*

Proof. For convenience, we write F_n, L_n, α, and β instead of $F_{k,n}$, $L_{k,n}$, α_k, and β_k, respectively. For (i), assume that $n \geq m$. By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} \lfloor F_n\alpha^m \rfloor &= \lfloor F_n\left(\alpha F_m + F_{m-1}\right) \rfloor \\ &= \lfloor \alpha F_m F_n \rfloor + F_{m-1}F_n \\ &= \lfloor \left(F_{n+1} - \beta^n\right) F_m \rfloor + F_{m-1}F_n \\ &= \lfloor -\beta^n F_m \rfloor + F_m F_{n+1} + F_{m-1}F_n \\ &= \lfloor -\beta^n F_m \rfloor + F_{n+m} \\ &= -\lfloor \beta^n F_m \rfloor - 1 + F_{n+m}. \end{aligned}$$

Thus, it suffices to show that $\lfloor \beta^n F_m \rfloor = [n \equiv 0 \pmod 2] - 1 = -[n \equiv 1 \pmod 2]$. To do so, we apply the Binet's formula throughout the following cases.

Case 1. n is even. In this case, we will show that $\lfloor \beta^n F_m \rfloor = 0$, i.e., $0 < \beta^n F_m < 1$.

Case 1.1. m is even. Then,

$$0 < \beta^n F_m \leq \beta^m F_m = \beta^m \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) = \frac{1 - \beta^{2m}}{\alpha - \beta} = \frac{1 - \beta^{2m}}{\sqrt{k^2+4}} < \frac{1}{\sqrt{k^2+4}} < 1.$$

Case 1.2. m is odd. Then, $n \geq m+1$ and

$$0 < \beta^n F_m \leq \beta^{m+1} F_m = \beta^{m+1} \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) = \frac{-\beta - \beta^{2m+1}}{\alpha - \beta} \leq \frac{-\beta - \beta^3}{\alpha - \beta} = \beta^2 < 1.$$

Case 2. n is odd. In this case, we will show that $\lfloor \beta^n F_m \rfloor = -1$, i.e., $-1 < \beta^n F_m < 0$.

Case 2.1. m is even. Then, $n \geq m+1$ and

$$0 > \beta^n F_m \geq \beta^{m+1} F_m = \beta^{m+1} \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) = \frac{\beta - \beta^{2m+1}}{\alpha - \beta} > \frac{\beta}{\alpha - \beta} > \beta > -1.$$

Case 2.2. m is odd. Then,

$$0 > \beta^n F_m \geq \beta^m F_m = \beta^m \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) = \frac{-1 - \beta^{2m}}{\alpha - \beta} \geq \frac{-1 - \beta^2}{\alpha - \beta} = \beta > -1.$$

For (ii), assume that $n \geq m+1$. By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we have that

$$\begin{aligned}\lfloor L_n\alpha^m \rfloor &= \lfloor L_n(\alpha F_m + F_{m-1})\rfloor \\ &= \lfloor \alpha F_m L_n \rfloor + F_{m-1}L_n \\ &= \lfloor \left(L_{n+1} + \beta^n\sqrt{k^2+4}\right)F_m\rfloor + F_{m-1}L_n \\ &= \lfloor \beta^n\sqrt{k^2+4}F_m\rfloor + F_mL_{n+1} + F_{m-1}L_n \\ &= \lfloor \beta^n\sqrt{k^2+4}F_m\rfloor + L_{n+m}.\end{aligned}$$

Thus, it suffices to show that $\lfloor \beta^n\sqrt{k^2+4}F_{k,m}\rfloor = -[n \equiv 1 \pmod 2]$. To do so, we investigate through the following cases.

Case 1. n is even. In this case, we will show that $\lfloor \beta^n\sqrt{k^2+4}F_m\rfloor = 0$, i.e., $0 < \beta^n\sqrt{k^2+4}F_m < 1$.

Case 1.1. m is even. Then,

$$0 < \beta^n\sqrt{k^2+4}F_m \leq \beta^m\sqrt{k^2+4}F_m = \beta^m\sqrt{k^2+4}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right) = 1-\beta^{2m} < 1.$$

Case 1.2. m is odd. Then,

$$\begin{aligned}0 < \beta^n\sqrt{k^2+4}F_m \leq \beta^{m+1}\sqrt{k^2+4}F_m &= \beta^{m+1}\sqrt{k^2+4}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right) \\ &= -\beta-\beta^{2m+1} \leq -\beta-\beta^3.\end{aligned}$$

If $k=1$, it is straightforward to check that $-\beta-\beta^3 < 1$. If $k \geq 2$, then $-\beta-\beta^3 < -2\beta = \sqrt{k^2+4}-k < \sqrt{k^2+2k+1}-k = 1$.

Case 2. n is odd. In this case, we will show that $\lfloor \beta^n\sqrt{k^2+4}F_m\rfloor = -1$, i.e., $-1 < \beta^n\sqrt{k^2+4}F_m < 0$.

Case 2.1. m is even. Then,

$$\begin{aligned}0 > \beta^n\sqrt{k^2+4}F_m \geq \beta^{m+1}\sqrt{k^2+4}F_m &= \beta^{m+1}\sqrt{k^2+4}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right) \\ &= \beta-\beta^{2m+1} > \beta > -1.\end{aligned}$$

Case 2.2. m is odd. Then, $n \geq m+2$. Using the same argument as in Case 1.2, we obtain

$$\begin{aligned}0 > \beta^n\sqrt{k^2+4}F_m \geq \beta^{m+2}\sqrt{k^2+4}F_m &= \beta^{m+2}\sqrt{k^2+4}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right) \\ &= -\beta^2-\beta^{2m+2} \geq -\beta^2-\beta^4 \\ &> \beta+\beta^3 > -1.\end{aligned}$$

This completes the proof. □

Corollary 2.7. *Let m and n be positive integers. Then, the following statements are valid.*

(i) $\{\alpha_k^n\} = -\beta_k^n + [n \equiv 0 \pmod 2]$.

(ii) *If $n \geq m$, then* $\{F_{k,n}\alpha_k^m\} = \dfrac{\beta_k^{n-m}}{\sqrt{k^2+4}}\left(\beta_k^{2m} - (-1)^m\right) + [n \equiv 0 \pmod 2]$.

(iii) $\{L_{k,n}\alpha_k^n\} = 1 - \beta_k^{2n}$.

(iv) *If $n > m$, then* $\{L_{k,n}\alpha_k^m\} = \beta_k^{n-m}\left((-1)^m - \beta_k^{2m}\right) + [n \equiv 1 \pmod 2]$.

Proof. Again, for simplicity, we write F_n, L_n, α, and β to denote $F_{k,n}$, $L_{k,n}$, α_k, and β_k, respectively.

(i) By Lemma 2.4, we obtain

$$\begin{aligned}\{\alpha^n\} &= \alpha^n - \lfloor \alpha^n \rfloor \\ &= \alpha^n - L_n + [n \equiv 0 \pmod 2] \\ &= \alpha^n - \alpha^n - \beta^n + [n \equiv 0 \pmod 2] \\ &= -\beta^n + [n \equiv 0 \pmod 2].\end{aligned}$$

(ii) By Lemma 2.6 (i), we obtain

$$\begin{aligned}\{F_n\alpha^m\} &= F_n\alpha^m - \lfloor F_n\alpha^m \rfloor \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta}\alpha^m - F_{n+m} + [n \equiv 0 \pmod 2] \\ &= \frac{\alpha^{n+m} - (-1)^m \beta^{n-m}}{\alpha - \beta} - \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} + [n \equiv 0 \pmod 2] \\ &= \frac{\beta^{n-m}}{\sqrt{k^2+4}}\left(\beta^{2m} - (-1)^m\right) + [n \equiv 0 \pmod 2].\end{aligned}$$

(iii) By Corollary 2.5, we obtain

$$\begin{aligned}\{L_n\alpha^n\} &= L_n\alpha^n - \lfloor L_n\alpha^n \rfloor \\ &= (\alpha^n + \beta^n)\,\alpha^n - L_{2n} + 2[n \equiv 1 \pmod 2] \\ &= \alpha^{2n} + (-1)^n - \alpha^{2n} - \beta^{2n} + 2[n \equiv 1 \pmod 2] \\ &= (-1)^n - \beta^{2n} + 2[n \equiv 1 \pmod 2] \\ &= 1 - \beta^{2n}.\end{aligned}$$

(iv) By Lemma 2.6 (ii), we obtain

$$\begin{aligned}\{L_n\alpha^m\} &= L_n\alpha^m - \lfloor L_n\alpha^m \rfloor \\ &= (\alpha^n + \beta^n)\,\alpha^m - L_{n+m} + [n \equiv 1 \pmod 2] \\ &= \alpha^{n+m} + (-1)^m \beta^{n-m} - \alpha^{n+m} - \beta^{n+m} + [n \equiv 1 \pmod 2] \\ &= \beta^{n-m}\left((-1)^m - \beta^{2m}\right) + [n \equiv 1 \pmod 2].\end{aligned}$$

□

3. The Difference Between Consecutive Terms in $B(\alpha_k^n)$

In [10], Kawsumarng, et al. studied the pattern of the difference between two consecutive terms in $B(\alpha_1)$ and $B(\alpha_1^2)$. Later, Pongsriiam [15] did the same for the sequence $B(\alpha_1^n)$, where $n \geq 3$. They also provided some properties of a certain segment in these sequences. Recall that a segment of a sequence $(x_n)_{n\geq 1}$ as a finite subsequence $(x_m, x_{m+1}, \ldots, x_{m+t})$, where both m and t are some positive integers. In this section, we further investigate the difference between two consecutive terms in $B(\alpha_k^n)$ for arbitrary positive integers k and n. Although, the idea used in [10] and [15] can be applied to obtain the results, we want to explore an alternative method. To do so, we considered $\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 0}$ as Sturmian languages in integers.

Before proceeding further, let us introduce all notations and theorems that will be used throughout this section. For each real number θ, let $a_0 = \lfloor \theta \rfloor$, $\theta_0 = \theta - a_0$, $a_n = \lfloor 1/\theta_{n-1} \rfloor$, and $\theta_n = (1/\theta_{n-1}) - a_n$ for all positive integers n. Then, the continued fraction expansion of θ is denoted by $[a_0, a_1, a_2, \ldots]$. A real number $\theta \in (0,1)$ is called a Sturm number if it is a quadratic irrational number with algebraic conjugate $\bar{\theta} \notin (0,1)$. The Sturmian sequence generated by a real number $\theta \in (0,1)$ is the sequence

$$c_\theta := (\lfloor (n+1)\,\theta \rfloor - \lfloor n\theta \rfloor)_{n\geq 1}.$$

The Sturmian sequence can be considered as the sequence of the differences between consecutive elements in $B(\theta)$. Clearly, c_θ is a sequence of zeros and ones. The sequence c_θ was studied by Komatsu and Poorten. They showed the following theorem.

Theorem 3.1. [11] *Let θ be an irrational number in the interval $(0,1)$. Then, there exists a morphism σ_θ on the alphabet $\{0,1\}$ such that $\sigma_\theta(c_\alpha) = c_\alpha$ if and only if (a) $\theta = [0, 1, a_2, \overline{a_3, \ldots, a_n}] > 1/2$ and $a_n \geq a_2$, when*

$$\sigma_\theta : 0 \mapsto T_{n-1}^{a_n - a_2} T_{n-2}, \quad 1 \mapsto T_{n-1},$$

or (b) $\theta = [0, a_1, \overline{a_2, \ldots, a_n}] < 1/2$ and $a_n + 1 \geq a_1 \geq 2$, when

$$\sigma_\theta : 0 \mapsto T_{n-1}, \quad 1 \mapsto T_{n-1}^{a_n+1-a_1} T_{n-2},$$

where $T_0 = 0$, $T_1 = 0^{a_1-1}1$, and $T_k = T_{k-1}^{a_k} T_{k-2}$ for all positive integer $k \geq 2$.

Later, Allouche and Dekking extended this result to a generalized Beatty sequence as shown in the following theorem.

Theorem 3.2. [1] *Let θ be a Sturm number. Let $V(\theta) := (p(\lfloor n\theta \rfloor) + qn + r)_{n\geq 1}$ and $f_\theta := (p(\lfloor (n+1)\,\theta \rfloor - \lfloor n\theta \rfloor) + q)_{n\geq 1}$, where p, q, and r are integers. Then, f_θ is the fixed point of σ_α on the alphabet $\{q, p+q\}$.*

Notice that f_θ is actually the sequence of the differences between consecutive elements in $V(\theta)$. From now on, let k be a positive integer.

Theorem 3.3. *Let n be an odd positive integer and let b be a nonnegative integer. Then,*

$$(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor)_{b\geq 1} = \lim_{\ell \to \infty} u_\ell,$$

where u_ℓ is the finite word $\sigma^\ell(L_n)$ when σ is the morphism on $\{L_{k,n}, L_{k,n}+1\}$ defined by

$$\sigma : L_{k,n} \mapsto (L_{k,n})^{L_{k,n}-1}(L_{k,n}+1), \quad L_{k,n}+1 \mapsto (L_{k,n})^{L_{k,n}-1}(L_{k,n}+1)(L_n).$$

Moreover, the following statements hold.

(i) *$\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor$ is equal to $L_{k,n}$ or $L_{k,n}+1$.*
(ii) *If $\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor = L_{k,n}+1$, then $\lfloor (b+2)\,\alpha_k^n \rfloor - \lfloor (b+1)\,\alpha_k^n \rfloor = L_{k,n}$.*
(iii) *$(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor)_{b\geq 1}$ does not contain the segment $(\underbrace{L_{k,n}, L_{k,n}, \ldots, L_{k,n}}_{(L_{k,n}+1)-\text{terms}})$.*

Proof. For convenience, we denote $F_{k,n}$, $L_{k,n}$, α_k, and β_k by F_n, L_n, α, and β, respectively. If $k=1$ and $n=1$, then it was shown in [1] that $(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor)_{b\geq 1}$ is the fixed point of the morphism σ_θ on $\{L_n, L_n+1\}$ defined by

$$\sigma_\theta : L_n \mapsto (L_n)^{L_n-1}(L_n+1), \quad L_n+1 \mapsto (L_n)^{L_n-1}(L_n+1)(L_n).$$

Assume that $k>1$ or $n>1$. Then, $L_n \geq 2$. Let $\theta := \alpha^n - L_n = -\beta^n \in (0, 1/2)$. Then, one can easily check that $\bar{\theta} = -(\bar{\beta})^n = -\alpha^n < -1$. Thus, θ is a Sturm number. Moreover, $B(\alpha^n) = (\lfloor n\theta \rfloor + nL_n)_{n\geq 1}$. According to Lemma 2.4 and Corollary 2.7 (i), we have that $\left\lfloor -\frac{1}{\beta^n} \right\rfloor =$

$\lfloor \alpha^n \rfloor = L_n$ and $\left\{ -\frac{1}{\beta^n} \right\} = \{\alpha^n\} = -\beta^n$. Thus, the continued expansion of θ is $\left[0, L_n, \overline{L_n}\right]$. As $L_n + 1 \geq L_n \geq 2$ for all $k \geq 2$, Theorem 3.1 and 3.2 imply that $\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 1}$ is the fixed point of the morphism σ_θ on $\{L_n, L_n + 1\}$ defined by

$$\sigma_\theta : L_n \mapsto (L_n)^{L_n - 1}\,(L_n + 1)\,, \quad L_n + 1 \mapsto (L_n)^{L_n - 1}\,(L_n + 1)\,(L_n)\,.$$

As the first letters of $\sigma_\theta(L_n)$ and $\sigma_\theta(L_n + 1)$ are the same, the morphism σ_θ has a unique fixed point (see [5] for details) that is $\lim_{\ell\to\infty} u_\ell$, where $u_\ell = \sigma_\theta^\ell(L_n)$. Therefore, $\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 1} = \lim_{\ell\to\infty} u_\ell$. Consequently, (i), (ii), and (iii) hold. □

Theorem 3.4. *Let n be an even positive integer and let b be a nonnegative integer. Then,*

$$\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 1} = \lim_{\ell\to\infty} u_\ell,$$

where u_ℓ is the finite word $\sigma^\ell(L_{k,n} - 1)$ when σ is the morphism on $\{L_{k,n} - 1, L_{k,n}\}$ defined by

$$\sigma : (L_{k,n} - 1) \mapsto (L_{k,n})^{L_{k,n} - 2}\,(L_{k,n} - 1)\,, \quad L_{k,n} \mapsto (L_{k,n})^{L_{k,n} - 2}\,(L_{k,n} - 1)\,(L_{k,n})\,.$$

Moreover, the following statements hold.

(i) *$\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor$ is equal to $L_{k,n} - 1$ or $L_{k,n}$.*
(ii) *If $\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor = L_{k,n} - 1$, then $\lfloor (b+2)\,\alpha_k^n \rfloor - \lfloor (b+1)\,\alpha_k^n \rfloor = L_{k,n}$.*
(iii) *$\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 0}$ does not contain the segment $(\underbrace{L_{k,n}, L_{k,n}, \ldots, L_{k,n}}_{L_{k,n}-\text{terms}})$.*

Proof. Again, we write F_n, L_n, α, and β for $F_{k,n}$, $L_{k,n}$, α_k, and β_k, respectively. Let $\theta := 1 - L_n + \alpha^n = 1 - \beta^n \in (1/2, 1)$. Then, one can easily check that $\bar{\theta} = 1 - \left(\bar{\beta}\right)^n = 1 - \alpha^n < 0$. Thus, θ is a Sturm number. Moreover, $B(\alpha^n) = \left(\lfloor n\theta \rfloor + (L_n)\,n - 1\right)_{n\geq 1}$. According to Lemma 2.4 and Corollary 2.7 (i), we have that $\left\lfloor \frac{1}{1-\beta^n} \right\rfloor = 1$, $\left\{ \frac{1}{1-\beta^n} \right\} = \frac{\beta^n}{1-\beta^n}$, $\left\lfloor \frac{1-\beta^n}{\beta^n} \right\rfloor = \lfloor \alpha^n - 1 \rfloor = L_n - 2$, and $\left\{ \frac{1-\beta^n}{\beta^n} \right\} = \{\alpha^n - 1\} = \{\alpha^n\} = 1 - \beta^n$. Thus, the continued expansion of θ is $\left[0, 1, L_n - 2, \overline{1, L_n - 2}\right]$. As $L_n - 2 \geq L_n - 2$, Theorem 3.1 and 3.2 imply that $\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 1}$ is the fixed point of the morphism σ_θ on $\{L_n - 1, L_n\}$ defined by

$$\sigma_\theta : (L_n - 1) \mapsto (L_n)^{L_n - 2}\,(L_n - 1)\,, \quad L_n \mapsto (L_n)^{L_n - 2}\,(L_n - 1)\,(L_n)\,.$$

Because the first letters of $\sigma_\theta(L_n - 1)$ and $\sigma_\theta(L_n)$ are the same, the morphism σ_θ has a unique fixed point that is $\lim_{\ell\to\infty} u_\ell$, where $u_\ell = \sigma_\theta^\ell(L_n - 1)$. Therefore, $\left(\lfloor (b+1)\,\alpha_k^n \rfloor - \lfloor b\alpha_k^n \rfloor\right)_{b\geq 1} = \lim_{\ell\to\infty} u_\ell$. Consequently, (i), (ii), and (iii) hold. □

4. k-Fibonacci and k-Lucas Numbers in $B(\alpha_k^n)$

In this section, we give necessary and sufficient conditions for $F_{k,n}$ and $L_{k,n}$ to be in $B(\alpha_k^m)$

Theorem 4.1. *Let m and n be positive integers such that $n \geq 2m$. Then, the following statements are true.*

(i) *If m is odd, then*
  (a) *$F_{k,n} \in B(\alpha_k^m)$ if and only if n is even, and*
  (b) *$F_{k,n} - 1 \in B(\alpha_k^m)$ if and only if n is odd.*
(ii) *If m is even, then*

(a) *$F_{k,n} \in B(\alpha_k^m)$ if and only if n is odd, and*
(b) *$F_{k,n} - 1 \in B(\alpha_k^m)$ if and only if n is even.*

Proof. For (i), assume that m is odd. If $k = 1$ and $m = 1$, then the statement is true as shown in [10, Theorem 3.2]. Thus, assume that $k \geq 2$ or $m \geq 3$. By Lemma 2.6 (i), if n is even, then

$$F_n = \lfloor F_{n-m}\alpha^m \rfloor + [n - m \equiv 0 \,(\mathrm{mod}\ 2)] = \lfloor F_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

Suppose that n is odd and there is a positive integer b such that $F_n = \lfloor b\alpha^m \rfloor$. Then by Lemma 2.6 (i), we obtain

$$\lfloor b\alpha^m \rfloor = \lfloor F_{n-m}\alpha^m \rfloor + [n - m \equiv 0 \,(\mathrm{mod}\ 2)] = \lfloor F_{n-m}\alpha^m \rfloor + 1.$$

By Theorem 3.3 (i), we obtain

$$1 = \lfloor b\alpha^m \rfloor - \lfloor F_{n-m}\alpha^m \rfloor \geq \lfloor (F_{n-m} + 1)\alpha^m \rfloor - \lfloor F_{n-m}\alpha^m \rfloor \geq L_m > 1,$$

which is a contradiction. So, (a) in (i) is proved.

Next, if n is odd, then Lemma 2.6 (i) implies that

$$F_n - 1 = \lfloor F_{n-m}\alpha^m \rfloor + [n - m \equiv 0 \,(\mathrm{mod}\ 2)] - 1 = \lfloor F_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

If n is even, then we obtain from (a) that $F_n \in B(\alpha^m)$ and we also know that the difference between distinct elements in $B(\alpha^m)$ is at least $L_m - [m \equiv 0 \,(\mathrm{mod}\ 2)] = L_m > 1$, and thus, $F_n - 1 \notin B(\alpha^m)$.

For (ii), assume that m is even. If $k = 1$ and $m = 2$, then the statement is true as shown in [10, Theorem 3.2]. Thus, assume that $k \geq 2$ or $m \geq 4$. By Lemma 2.6 (i), if n is odd, then

$$F_n = \lfloor F_{n-m}\alpha^m \rfloor + [n - m \equiv 0 \,(\mathrm{mod}\ 2)] = \lfloor F_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

Moreover, if n is even and there is a positive integer b such that $F_n = \lfloor b\alpha^m \rfloor$, then by Lemma 2.6 (i), we obtain

$$\lfloor b\alpha^m \rfloor = \lfloor F_{n-m}\alpha^m \rfloor + [n - m \equiv 0 \,(\mathrm{mod}\ 2)] = \lfloor F_{n-m}\alpha^m \rfloor + 1,$$

which contradicts Theorem 3.3 (i) because

$$1 = \lfloor b\alpha^m \rfloor - \lfloor F_{n-m}\alpha^m \rfloor \geq \lfloor (F_{n-m} + 1)\alpha^m \rfloor - \lfloor F_{n-m}\alpha^m \rfloor \geq L_m - 1 > 1.$$

This proves (a) in (ii).

Next, if n is even, then

$$F_n - 1 = \lfloor F_{n-m}\alpha^m \rfloor + [n - m \equiv 0 \,(\mathrm{mod}\ 2)] - 1 = \lfloor F_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

If n is odd, then we obtain from (a) that $F_n \in B(\alpha^m)$ and we also know that the difference between distinct elements in $B(\alpha^m)$ is at least $L_m - [m \equiv 0 \,(\mathrm{mod}\ 2)] = L_m - 1 > 1$, and thus, $F_n - 1 \notin B(\alpha^m)$. This completes the proof. □

Recall that Beatty's theorem implies that $B(\alpha_1)$ and $B\left(\alpha_1^2\right)$ form a partition of $\mathbb{N}$. By applying Beatty's theorem [2, 3] and using a similar argument as in [10, Theorem 3.2], we obtain the following result for 1-Lucas numbers.

Proposition 4.2. *Let n be a positive integer. Then, the following statements are true.*

(i) *For $n \geq 3$, $L_{1,n} \in B(\alpha_1)$ if and only if n is odd.*
(ii) *For $n \geq 4$, $L_{1,n} - 1 \in B(\alpha_1)$ if and only if n is even.*
(iii) *For $n \geq 6$, $L_{1,n} \in B\left(\alpha_1^2\right)$ if and only if n is even.*
(iv) *For $n \geq 5$, $L_{1,n} - 1 \in B\left(\alpha_1^2\right)$ if and only if n is odd.*

Proof. When $n = 3$ or 4, the results can be easily checked. So, assume throughout that $n \geq 5$. By Lemma 2.6 (ii), we obtain

$$L_{1,n} - [n \equiv 0 \,(\mathrm{mod}\ 2)] = \lfloor L_{1,n-1}\alpha_1 \rfloor \in B(\alpha_1),$$

$$L_{1,n} - [n \equiv 1 \,(\mathrm{mod}\ 2)] = \lfloor L_{1,n-2}\alpha_1^2 \rfloor \in B\left(\alpha_1^2\right).$$

If n is odd, then above equalities yield $L_{1,n} \in B(\alpha_1)$ and $L_{1,n} - 1 \in B\left(\alpha_1^2\right)$. Thus, by Beatty's theorem, we have $L_{1,n} \notin B\left(\alpha_1^2\right)$ and $L_{1,n} - 1 \notin B(\alpha_1)$.

On the other hand, if n is even, then the above equalities yield $L_{1,n} - 1 \in B(\alpha_1)$ and $L_{1,n} \in B\left(\alpha_1^2\right)$. Again, by Beatty's theorem, we have $L_{1,n} - 1 \notin B\left(\alpha_1^2\right)$ and $L_{1,n} \notin B(\alpha_1)$. □

Theorem 4.3. *Let m and n be positive integers such that $n \geq 2m + 2$. Then, the following statements are true.*

- *(i) If m is odd, then*
  - *(a) $L_{k,n} \in B(\alpha_k^m)$ if and only if n is odd, and*
  - *(b) $L_{k,n} - 1 \in B(\alpha_k^m)$ if and only if n is even.*
- *(ii) If m is even, then*
  - *(a) $L_{k,n} \in B(\alpha_k^m)$ if and only if n is even, and*
  - *(b) $L_{k,n} - 1 \in B(\alpha_k^m)$ if and only if n is odd.*

Proof. For (i), assume that m is odd. If $k = 1$ and $m = 1$, then the statement is valid by Proposition 4.2. Thus, assume that $k \geq 2$ or $m \geq 3$. By Lemma 2.6 (ii), if n is odd, then

$$L_n = \lfloor L_{n-m}\alpha^m \rfloor + [n - m \equiv 1 \,(\mathrm{mod}\ 2)] = \lfloor L_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

Because $L_n \in B(\alpha^m)$ and the difference between distinct elements of $B(\alpha^m)$ is larger than 1, we see that $L_n - 1 \notin B(\alpha^m)$.

On the other hand, if n is even, then Lemma 2.6 (ii) implies that

$$L_n - 1 = \lfloor L_{n-m}\alpha^m \rfloor + [n - m \equiv 1 \,(\mathrm{mod}\ 2)] - 1 = \lfloor L_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

Because $L_n - 1 \in B(\alpha^m)$ and the difference between distinct elements of $B(\alpha^m)$ is larger than 1, we see that $L_n \notin B(\alpha^m)$. This proves (i).

For (ii), assume that m is even. If $k = 1$ and $m = 2$, then the statement is valid by Proposition 4.2. Thus, assume that $k \geq 2$ or $m \geq 4$. By Lemma 2.6 (ii), if n is even, then

$$L_n = \lfloor L_{n-m}\alpha^m \rfloor + [n - m \equiv 1 \,(\mathrm{mod}\ 2)] = \lfloor L_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

Because $L_n \in B(\alpha^m)$ and the difference between distinct elements of $B(\alpha^m)$ is larger than 1, we see that $L_n - 1 \notin B(\alpha^m)$. On the other hand, if n is odd, then

$$L_n - 1 = \lfloor L_{n-m}\alpha^m \rfloor + [n - m \equiv 1 \,(\mathrm{mod}\ 2)] - 1 = \lfloor L_{n-m}\alpha^m \rfloor \in B(\alpha^m).$$

Because $L_n - 1 \in B(\alpha^m)$ and the difference between distinct elements of $B(\alpha^m)$ is larger than 1, we see that $L_n \notin B(\alpha^m)$. This proves (ii). □

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