ADDITIONAL SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES

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Abstract. We investigate four infinite sums involving gibonacci polynomial squares and their numeric versions, and deduce their Pell versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [4]. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1)$ and $2Q_n = q_n(1)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $E = \sqrt{x^2+1}$.

It follows by the Binet-like formulas that $\lim\limits_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$ [4, 5, 6].

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [4, 5]; they follow by the Binet-like formulas:

$$f_{2n} = f_n l_n; \tag{1}$$

$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n; \tag{2}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{3}$$

$$\Delta^2\left(f_{n+k}^4 - f_{n-k}^4\right) = f_{4k}f_{4n} - 4(-1)^{n+k} f_{2k}f_{2n}; \tag{4}$$

$$l_{n+k}^4 - l_{n-k}^4 = \Delta^2[f_{4k}f_{4n} + 4(-1)^{n+k} f_{2k}f_{2n}]. \tag{5}$$

1.2. Telescoping Gibonacci Sums.

In our investigation of infinite sums of gibonacci polynomial squares in [6], we employed the following telescoping sums, where the degree λ of each g_n is two; for convenience, we call them lemmas:

Lemma 1.

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}^{\lambda}} - \frac{1}{g_{2n+k}^{\lambda}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}^{\lambda}}.$$

Lemma 2.
$$\sum_{\substack{n=k/2+1\\k\geq 2,\, even}}^{\infty}\left(\frac{1}{g_{2n-k}^{\lambda}}-\frac{1}{g_{2n+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}^{\lambda}}.$$

Lemma 3.
$$\sum_{\substack{n=(k+1)/2\\k\geq 1,\, odd}}^{\infty}\left(\frac{1}{g_{2n+1-k}^{\lambda}}-\frac{1}{g_{2n+1+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}^{\lambda}}.$$

Lemma 4.
$$\sum_{\substack{n=k/2\\k\geq 2,\, even}}^{\infty}\left(\frac{1}{g_{2n+1-k}^{\lambda}}-\frac{1}{g_{2n+1+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r-1}^{\lambda}}.$$

We now restrict our discourse to applications of these lemmas with $\lambda = 4$. The aforementioned identities, coupled with these lemmas, play a pivotal role in our explorations.

2. Sums Involving Gibonacci Polynomial Squares

In the interest of brevity, we let

$$L = \begin{cases}(k+1)/2, k\geq 1, & \text{if } k \text{ is odd;}\\ k/2+1, k\geq 2, & \text{otherwise;}\end{cases} \qquad s = \begin{cases}2r-1, & \text{if } k \text{ is odd;}\\ 2r, & \text{otherwise;}\end{cases}$$
$$M = \begin{cases}(k+1)/2, k\geq 1, & \text{if } k \text{ is odd;}\\ k/2, k\geq 2, & \text{otherwise;}\end{cases} \quad\text{and}\quad t = \begin{cases}2r, & \text{if } k \text{ is odd;}\\ 2r-1, & \text{otherwise,}\end{cases}$$

where $1 \leq r \leq k$. We now begin our explorations with the following result.

Theorem 1. *Let k be a positive integer. Then*
$$\sum_{n=L}^{\infty}\frac{f_{4k}f_{8n}-4(-1)^k f_{2k}f_{4n}}{\left[f_{2n}^2-(-1)^k f_k^2\right]^4}=\Delta^2\sum_{r=1}^{k}\frac{1}{f_s^4}.$$

Proof. Using identities (3) and (4), we have
$$\begin{aligned}\frac{f_{4k}f_{8n}-4(-1)^k f_{2k}f_{4n}}{\Delta^2\left[f_{2n}^2-(-1)^k f_k^2\right]^4} &= \frac{f_{2n+k}^4-f_{2n-k}^4}{f_{2n+k}^4 f_{2n-k}^4}\\ &= \frac{1}{f_{2n-k}^4}-\frac{1}{f_{2n+k}^4}. \qquad (6)\end{aligned}$$

Suppose k is odd. With Lemma 1, this yields
$$\sum_{\substack{n=(k+1)/2\\k\geq 1,\, odd}}^{\infty}\frac{f_{4k}f_{8n}+4f_{2k}f_{4n}}{\left(f_{2n}^2+f_k^2\right)^4}=\sum_{r=1}^{k}\frac{\Delta^2}{f_{2r-1}^4}. \qquad (7)$$

On the other hand, suppose k is even. Then, by Lemma 2, equation (6) gives
$$\sum_{\substack{n=k/2+1\\k\geq 2,\, even}}^{\infty}\frac{f_{4k}f_{8n}-4f_{2k}f_{4n}}{\left(f_{2n}^2-f_k^2\right)^4}=\sum_{r=1}^{k}\frac{\Delta^2}{f_{2r}^4}. \qquad (8)$$

Combining the two cases, we get the desired result. □

In particular, we get

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n} + 4f_2 f_{4n}}{\left(f_{2n}^2 + 1\right)^4} = \Delta^2; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n} + 4F_{4n}}{\left(F_{2n}^2 + 1\right)^4} = 5;$$

$$\sum_{n=2}^{\infty} \frac{f_8 f_{8n} - 4f_4 f_{4n}}{\left(f_{2n}^2 - x^2\right)^4} = \frac{\Delta^2\left(f_2^4 + f_4^4\right)}{f_2^4 f_4^4}; \qquad \sum_{n=2}^{\infty} \frac{7F_{8n} - 4F_{4n}}{\left(F_{2n}^2 - 1\right)^4} = \frac{410}{243}.$$

With identity (2), it follows from Theorem 1 that

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} - 4(-1)^k f_{2k} f_{4n}}{\left[l_{2n}^2 - (-1)^k \Delta^2 f_k^2 - 4\right]^4} = \frac{1}{\Delta^6} \sum_{r=1}^{k} \frac{1}{f_s^4}. \tag{9}$$

This implies

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n} + 4f_2 f_{4n}}{\left(l_{2n}^2 + x^2\right)^4} = \frac{1}{\Delta^6}; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n} + 4F_{4n}}{\left(L_{2n}^2 + 1\right)^4} = \frac{1}{125};$$

$$\sum_{n=2}^{\infty} \frac{f_8 f_{8n} - 4f_4 f_{4n}}{\left[l_{2n}^2 - (x^2+2)^2\right]^4} = \frac{f_2^4 + f_4^4}{\Delta^6 f_2^4 f_4^4}; \qquad \sum_{n=2}^{\infty} \frac{7F_{8n} - 4F_{4n}}{\left(L_{2n}^2 - 9\right)^4} = \frac{82}{30,375}.$$

Next, we explore the Lucas version of Theorem 2.

Theorem 2. *Let k be a positive integer. Then*

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} + 4(-1)^k f_{2k} f_{4n}}{\left[l_{2n}^2 + (-1)^k \Delta^2 f_k^2\right]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_s^4}.$$

Proof. Using equations (3) and (5), we get

$$\begin{aligned} \frac{\Delta^2\left[f_{4k} f_{8n} + 4(-1)^k f_{2k} f_{4n}\right]}{\left[l_{2n}^2 + (-1)^k \Delta^2 f_k^2\right]^4} &= \frac{l_{2n+k}^4 - l_{2n-k}^4}{l_{2n+k}^4 l_{2n-k}^4} \\ &= \frac{1}{l_{2n-k}^4} - \frac{1}{l_{2n+k}^4}. \end{aligned} \tag{10}$$

Let k be odd. By Lemma 1, this yields

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{4k} f_{8n} - 4f_{2k} f_{4n}}{\left(l_{2n}^2 - \Delta^2 f_k^2\right)^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_{2r-1}^4}. \tag{11}$$

When k is even, equation (10), coupled with Lemma 2, gives

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{f_4 f_{8n} + 4f_{2k} f_{4n}}{\left(l_{2n}^2 + \Delta^2 f_k^2\right)^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_{2r}^2}. \tag{12}$$

Combining the two cases yields the given result, as desired. □

It follows from equations (11) and (12) that

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n} - 4f_2 f_{4n}}{\left(l_{2n}^2 - \Delta^2\right)^4} = \frac{1}{\Delta^2 x^4}; \qquad \sum_{n=2}^{\infty} \frac{3F_{8n} - 4F_{4n}}{\left(L_{2n}^2 - 5\right)^4} = \frac{1}{5};$$

$$\sum_{n=2}^{\infty} \frac{f_8 f_{8n} + 4f_4 f_{4n}}{\left(l_{2n}^2 + \Delta^2 x^2\right)^4} = \frac{l_2^4 + l_4^4}{\Delta^2 l_2^4 l_4^4}; \qquad \sum_{n=2}^{\infty} \frac{7F_{8n} + 4F_{4n}}{\left(L_{2n}^2 + 5\right)^4} = \frac{2,482}{2,917,215}.$$

Using identity (2), it follows from the theorem that

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} + 4(-1)^k f_{2k} f_{4n}}{\left[\Delta^2 f_{2n}^2 + (-1)^k \Delta^2 f_k^2 + 4\right]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_s^4}.$$

This implies

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n} - 4 f_2 f_{4n}}{\left(\Delta^2 f_{2n}^2 - x^2\right)^4} = \frac{1}{\Delta^2 x^4}; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n} - 4F_{4n}}{(5F_{2n}^2 - 1)^4} = \frac{1}{5};$$

$$\sum_{n=2}^{\infty} \frac{f_8 f_{8n} + 4 f_4 f_{4n}}{\left[\Delta^2 f_{2n}^2 + (x^2+2)^2\right]^4} = \frac{l_2^4 + l_4^4}{\Delta^2 l_2^4 l_4^4}; \qquad \sum_{n=2}^{\infty} \frac{7F_{8n} + 4F_{4n}}{\left(5F_{2n}^2 + 9\right)^4} = \frac{2,482}{2,917,215}.$$

The next result is also an application of identities (3) and (4).

Theorem 3. *Let k be a positive integer. Then*

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} + 4(-1)^k f_{2k} f_{4n+2}}{\left[f_{2n+1}^2 + (-1)^k f_k^2\right]^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_t^4}.$$

Proof. With identities (3) and (4), we get

$$\frac{f_{4k} f_{8n+4} + 4(-1)^k f_{2k} f_{4n+2}}{\Delta^2 \left[f_{2n+1}^2 + (-1)^k f_k^2\right]^4} = \frac{f_{2n+1+k}^4 - f_{2n+1-k}^4}{f_{2n+1+k}^4 f_{2n+1-k}^4}$$

$$= \frac{1}{f_{2n+1-k}^4} - \frac{1}{f_{2n+1+k}^4}. \tag{13}$$

Now, let k be odd. With Lemma 3, this yields

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{4k} f_{8n+4} - 4 f_{2k} f_{4n+2}}{\left(f_{2n+1}^2 - f_k^2\right)^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_{2r}^4}. \tag{14}$$

With k even and Lemma 4, it follows from equation (13) that

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{f_{4k} f_{8n+4} + 4 f_{2k} f_{4n+2}}{\left(f_{2n+1}^2 + f_k^2\right)^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_{2r-1}^4}. \tag{15}$$

Equation (14), together with equation (15), gives the desired result. □

In particular, we have

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n+4} - 4 f_2 f_{4n+2}}{\left(f_{2n+1}^2 - 1\right)^4} = \frac{\Delta^2}{x^4}; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n+4} - 4F_{4n+2}}{(F_{2n+1}^2 - 1)^4} = 5;$$

$$\sum_{n=1}^{\infty} \frac{f_8 f_{8n+4} + 4 f_4 f_{4n+2}}{\left(f_{2n+1}^2 + x^2\right)^4} = \frac{\Delta^2 \left(f_1^4 + f_3^4\right)}{f_3^4}; \qquad \sum_{n=1}^{\infty} \frac{7F_{8n+4} + 4F_{4n+2}}{\left(F_{2n+1}^2 + 1\right)^4} = \frac{85}{48}.$$

Using identity (2), Theorem 3 yields

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} + 4(-1)^k f_{2k} f_{4n+2}}{\left[l_{2n+1}^2 + (-1)^k \Delta^2 f_k^2 + 4\right]^4} = \frac{1}{\Delta^6} \sum_{r=1}^{k} \frac{1}{f_t^4}.$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n+4} - 4f_2 f_{4n+2}}{\left(l_{2n+1}^2 - x^2\right)^4} = \frac{1}{\Delta^6 x^4}; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n+4} - 4F_{4n+2}}{(L_{2n+1}^2 - 1)^4} = \frac{1}{125};$$

$$\sum_{n=1}^{\infty} \frac{f_8 f_{8n+4} + 4f_4 f_{4n+2}}{\left[l_{2n+1}^2 + (x^2+2)^2\right]^4} = \frac{f_1^4 + f_3^4}{\Delta^6 f_3^4}; \qquad \sum_{n=1}^{\infty} \frac{7F_{8n+4} + 4F_{4n+2}}{\left(L_{2n+1}^2 + 9\right)^4} = \frac{17}{6,000}.$$

The following result presents the Lucas counterpart of Theorem 3.

Theorem 4. *Let k be a positive integer. Then*

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} - 4(-1)^k f_{2k} f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_t^4}.$$

Proof. Using identities (3) and (5), we have

$$\begin{aligned} \frac{\Delta^2 \left[f_{4k} f_{8n+4} - 4(-1)^k f_{2k} f_{4n+2}\right]}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^4} &= \frac{l_{2n+1+k}^4 - l_{2n+1-k}^4}{l_{2n+1+k}^4 l_{2n+1-k}^4} \\ &= \frac{1}{l_{2n+1-k}^4} - \frac{1}{l_{2n+1+k}^4}. \end{aligned} \tag{16}$$

With k odd, and Lemma 3, this yields

$$\sum_{\substack{n=(k+1)/2 \\ k\ge 1,\, odd}}^{\infty} \frac{f_{4k} f_{8n+4} + 4f_{2k} f_{4n+2}}{\left(l_{2n+1}^2 + \Delta^2 f_k^2\right)^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_{2r}^4}. \tag{17}$$

When k is even, using Lemma 4, equation (16) yields

$$\sum_{\substack{n=k/2 \\ k\ge 2,\, even}}^{\infty} \frac{f_{4k} f_{8n+4} - 4f_{2k} f_{4n+2}}{\left(l_{2n+1}^2 - \Delta^2 f_k^2\right)^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_{2r-1}^4}. \tag{18}$$

The given result now follows by equations (17) and (18), as desired. □

It follows from this theorem that

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n+4} + 4f_2 f_{4n+2}}{\left(l_{2n+1}^2 + \Delta^2\right)^4} = \frac{1}{\Delta^2 l_2^4}; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n+4} + 4F_{4n+2}}{(L_{2n+1}^2 + 5)^4} = \frac{1}{405};$$

$$\sum_{n=1}^{\infty} \frac{f_8 f_{8n+4} - 4f_4 f_{4n+2}}{\left(l_{2n+1}^2 - \Delta^2 x^2\right)^4} = \frac{l_1^4 + l_3^4}{\Delta^2 l_1^4 l_3^4}; \qquad \sum_{n=1}^{\infty} \frac{7F_{8n+4} - 4F_{4n+2}}{\left(L_{2n+1}^2 - 5\right)^4} = \frac{257}{3,840}.$$

Using identity (2), Theorem 4 yields

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} - 4(-1)^k f_{2k} f_{4n+2}}{\left[\Delta^2 f_{2n+1}^2 - (-1)^k \Delta^2 f_k^2 - 4\right]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_t^4}.$$

This yields

$$\sum_{n=1}^{\infty} \frac{f_4 f_{8n+4} + 4f_2 f_{4n+2}}{\left(\Delta^2 f_{2n+1}^2 + x^2\right)^4} = \frac{1}{\Delta^2 l_2^4}; \qquad \sum_{n=1}^{\infty} \frac{3F_{8n+4} + 4F_{4n+2}}{(5F_{2n+1}^2 + 1)^4} = \frac{1}{405};$$

$$\sum_{n=1}^{\infty} \frac{f_8 f_{8n+4} - 4f_4 f_{4n+2}}{\left[\Delta^2 f_{2n+1}^2 - (x^2+2)^2\right]^4} = \frac{l_1^4 + l_3^4}{\Delta^2 l_1^4 l_3^4}; \qquad \sum_{n=1}^{\infty} \frac{7F_{8n+4} - 4F_{4n+2}}{\left(5F_{2n+1}^2 - 9\right)^4} = \frac{257}{3,840}.$$

3. Pell Consequences

With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, Theorems 1–4 yield the following Pell versions:

$$\begin{aligned}
\sum_{n=L}^{\infty} \frac{p_{4k}p_{8n} - 4(-1)^k p_{2k}p_{4n}}{\left[p_{2n}^2 - (-1)^k p_k^2\right]^4} &= 4E^2 \sum_{r=1}^{k} \frac{1}{p_s^4};\\
\sum_{n=L}^{\infty} \frac{p_{4k}p_{8n} + 4(-1)^k p_{2k}p_{4n}}{\left[q_{2n}^2 + 4(-1)^k E^2 p_k^2\right]^4} &= \frac{1}{4E^2} \sum_{r=1}^{k} \frac{1}{q_s^4};\\
\sum_{n=M}^{\infty} \frac{p_{4k}p_{8n+4} + 4(-1)^k p_{2k}p_{4n+2}}{\left[p_{2n+1}^2 + (-1)^k p_k^2\right]^4} &= 4E^2 \sum_{r=1}^{k} \frac{1}{p_t^4};\\
\sum_{n=M}^{\infty} \frac{p_{4k}p_{8n+4} - 4(-1)^k p_{2k}p_{4n+2}}{\left[q_{2n+1}^2 - 4(-1)^k E^2 p_k^2\right]^4} &= \frac{1}{4E^2} \sum_{r=1}^{k} \frac{1}{q_t^4},
\end{aligned}$$

respectively. Consequently, we have

$$\begin{aligned}
\sum_{n=L}^{\infty} \frac{P_{4k}P_{8n} - 4(-1)^k P_{2k}P_{4n}}{\left[P_{2n}^2 - (-1)^k P_k^2\right]^4} &= 8 \sum_{r=1}^{k} \frac{1}{P_s^4};\\
\sum_{n=L}^{\infty} \frac{P_{4k}P_{8n} + 4(-1)^k P_{2k}P_{4n}}{\left[Q_{2n}^2 + 2(-1)^k P_k^2\right]^4} &= 2 \sum_{r=1}^{k} \frac{1}{Q_s^4};\\
\sum_{n=M}^{\infty} \frac{P_{4k}P_{8n+4} + 4(-1)^k P_{2k}P_{4n+2}}{\left[P_{2n+1}^2 + (-1)^k P_k^2\right]^4} &= 8 \sum_{r=1}^{k} \frac{1}{P_t^4};\\
\sum_{n=M}^{\infty} \frac{P_{4k}P_{8n+4} - 4(-1)^k P_{2k}P_{4n+2}}{\left[Q_{2n+1}^2 - 2(-1)^k P_k^2\right]^4} &= 2 \sum_{r=1}^{k} \frac{1}{Q_t^4},
\end{aligned}$$

again, respectively.

4. Chebyshev and Vieta Implications

Finally, we add that Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$, where $i = \sqrt{-1}$ [2, 3, 4]. They can be employed to find the Chebyshev and Vieta versions of Theorems 1–4. In the interest of brevity, we omit them and encourage gibonacci enthusiasts to explore them.

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