SUMS INVOLVING TWO CLASSES OF GIBONACCI POLYNOMIALS

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Abstract. We explore six infinite sums involving gibonacci polynomial squares and their implications.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $2\alpha = x + \Delta$.

It follows by the Binet-like formulas that $\lim\limits_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$ and $\lim\limits_{m\to\infty} \dfrac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 3]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1.1}$$

$$g_{n+k+1}g_{n-k} - g_{n+k}g_{n-k+1} = \begin{cases} (-1)^{n+k+1} f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_{2k}, & \text{otherwise.} \end{cases} \tag{1.2}$$

These properties can be confirmed using the Binet-like formulas.

2. Telescoping Gibonacci Sums

The following lemmas present telescoping gibonacci sums. While recursion plays an important role in all of them, the lemmas play a major role in our explorations.

Lemma 2.1. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(2n-1)k+1}}{g^{\lambda}_{(2n-1)k}} - \frac{g^{\lambda}_{(2n+1)k+1}}{g^{\lambda}_{(2n+1)k}} \right] = \frac{g^{\lambda}_{k+1}}{g^{\lambda}_{k}} - \alpha^{\lambda}. \tag{2.1}$$

Proof. Using recursion [2], we will first establish that

$$\sum_{n=1}^{m} \left[\frac{g^{\lambda}_{(2n-1)k+1}}{g^{\lambda}_{(2n-1)k}} - \frac{g^{\lambda}_{(2n+1)k+1}}{g^{\lambda}_{(2n+1)k}} \right] = \frac{g^{\lambda}_{k+1}}{g^{\lambda}_{k}} - \frac{g^{\lambda}_{(2m+1)k+1}}{g^{\lambda}_{(2m+1)k}}. \tag{2.2}$$

Letting A_m denote the left-hand side (LHS) of this equation and B_m its right-hand side (RHS), we get

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

With recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

This confirms the validity of equation (2.2).

Because $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$, equation (2.2) yields the desired result. □

The following lemma can be established using the same steps as above. So, in the interest of conciseness, we omit its proof.

Lemma 2.2. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g_{2nk+1}^{\lambda}}{g_{2nk}^{\lambda}} - \frac{g_{(2n+2)k+1}^{\lambda}}{g_{(2n+2)k}^{\lambda}} \right] = \frac{g_{2k+1}^{\lambda}}{g_{2k}^{\lambda}} - \alpha^{\lambda}. \tag{2.3}$$

The next four lemmas involve telescoping sums involving a gibonacci polynomial of a different class.

Lemma 2.3. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g_{(4n-3)k+1}^{\lambda}}{g_{(4n-3)k}^{\lambda}} - \frac{g_{(4n+1)k+1}^{\lambda}}{g_{(4n+1)k}^{\lambda}} \right] = \frac{g_{k+1}^{\lambda}}{g_{k}^{\lambda}} - \alpha^{\lambda}. \tag{2.4}$$

Proof. With recursion [2], we will first prove that

$$\sum_{n=1}^{m} \left[\frac{g_{(4n-3)k+1}^{\lambda}}{g_{(4n-3)k}^{\lambda}} - \frac{g_{(4n+1)k+1}^{\lambda}}{g_{(4n+1)k}^{\lambda}} \right] = \frac{g_{k+1}^{\lambda}}{g_{k}^{\lambda}} - \frac{g_{(4m+1)k+1}^{\lambda}}{g_{(4m+1)k}^{\lambda}}. \tag{2.5}$$

By letting A_m = LHS of this equation and B_m its RHS, we get

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

Using recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

confirming equation (2.5).

The given result now follows from it, as desired. □

We can confirm the next three lemmas using the same technique. Consequently, we omit their proofs also.

Lemma 2.4. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g_{(4n-2)k+1}^{\lambda}}{g_{(4n-2)k}^{\lambda}} - \frac{g_{(4n+2)k+1}^{\lambda}}{g_{(4n+2)k}^{\lambda}} \right] = \frac{g_{2k+1}^{\lambda}}{g_{2k}^{\lambda}} - \alpha^{\lambda}. \tag{2.6}$$

Lemma 2.5. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g_{(4n-1)k+1}^{\lambda}}{g_{(4n-1)k}^{\lambda}} - \frac{g_{(4n+3)k+1}^{\lambda}}{g_{(4n+3)k}^{\lambda}} \right] = \frac{g_{3k+1}^{\lambda}}{g_{3k}^{\lambda}} - \alpha^{\lambda}. \tag{2.7}$$

Lemma 2.6. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty}\left[\frac{g_{4nk+1}^{\lambda}}{g_{4nk}^{\lambda}}-\frac{g_{(4n+4)k+1}^{\lambda}}{g_{(4n+4)k}^{\lambda}}\right]=\frac{g_{4k+1}^{\lambda}}{g_{4k}^{\lambda}}-\alpha^{\lambda}. \tag{2.8}$$

3. Gibonacci Sums

With identities (1.1) and (1.2), and the lemmas at our disposal, we are now ready for further explorations, with the restriction that $\lambda = 1$. In the interest of brevity, we now let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu^* = \begin{cases} 1, & \text{if } g_n = f_n; \\ -1, & \text{otherwise.} \end{cases}$$

The first result invokes Lemma 2.1.

Theorem 3.1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{(-1)^k\mu\nu^* f_{2k}}{g_{2nk}^2-(-1)^k\mu\nu^* f_k^2}=\frac{g_{k+1}}{g_k}-\alpha. \tag{3.1}$$

Proof. Suppose $g_n = f_n$. Lemma 2.1, coupled with identities (1.1) and (1.2), then yields

$$\begin{aligned}
\frac{(-1)^k f_{2k}}{f_{2nk}^2-(-1)^k f_k^2} &= \frac{f_{(2n+1)k}f_{(2n-1)k+1}-f_{(2n+1)k+1}f_{(2n-1)k}}{f_{(2n+1)k}f_{(2n-1)k}};\\
\sum_{n=1}^{\infty}\frac{(-1)^k f_{2k}}{f_{2nk}^2-(-1)^k f_k^2} &= \sum_{n=1}^{\infty}\left[\frac{f_{(2n-1)k+1}}{f_{(2n-1)k}}-\frac{f_{(2n+1)k+1}}{f_{(2n+1)k}}\right]\\
&= \frac{f_{k+1}}{f_k}-\alpha.
\end{aligned} \tag{3.2}$$

On the other hand, let $g_n = l_n$. Using identities (1.1) and (1.2), and Lemma 2.1, we get

$$\begin{aligned}
\frac{(-1)^{k+1}\Delta^2 f_{2k}}{l_{2nk}^2+(-1)^k\Delta^2 f_k^2} &= \frac{l_{(2n+1)k}l_{(2n-1)k+1}-l_{(2n+1)k+1}l_{(2n-1)k}}{l_{(2n+1)k}l_{(2n-1)k}};\\
\sum_{n=1}^{\infty}\frac{(-1)^{k+1}\Delta^2 f_{2k}}{l_{2nk}^2+(-1)^k\Delta^2 f_k^2} &= \sum_{n=1}^{\infty}\left[\frac{l_{(2n-1)k+1}}{l_{(2n-1)k}}-\frac{l_{(2n-1)k+1}}{l_{(2n-1)k}}\right]\\
&= \frac{l_{k+1}}{l_k}-\alpha.
\end{aligned} \tag{3.3}$$

Combining equations (3.2) and (3.3), we get the desired result. □

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{F_{2n}^2+1} &= -\frac{1}{2}+\frac{\sqrt{5}}{2}; & \sum_{n=1}^{\infty}\frac{1}{L_{2n}^2-5} &= \frac{1}{2}-\frac{\sqrt{5}}{10};\\
\sum_{n=1}^{\infty}\frac{1}{F_{4n}^2-1} &= \frac{1}{2}-\frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty}\frac{1}{L_{4n}^2+5} &= -\frac{1}{18}+\frac{\sqrt{5}}{30};\\
\sum_{n=1}^{\infty}\frac{1}{F_{6n}^2+4} &= -\frac{1}{8}+\frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty}\frac{1}{L_{6n}^2-20} &= \frac{1}{32}-\frac{\sqrt{5}}{80}.
\end{aligned}$$

The next result invokes Lemma 2.2.

Theorem 3.2. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\nu^* f_{2k}}{g_{(2n+1)k}^2 - \mu\nu^* f_k^2} = \frac{g_{2k+1}}{g_{2k}} - \alpha. \tag{3.4}$$

Proof. Let $g_n = f_n$. Using identities (1.1) and (1.2), Lemma 2.2 yields

$$\begin{aligned} \frac{f_{2k}}{f_{(2n+1)k}^2 - f_k^2} &= \frac{f_{(2n+2)k} f_{2nk+1} - f_{(2n+2)k+1} f_{2nk}}{f_{(2n+2)k} f_{2nk}}; \\ \sum_{n=1}^{\infty} \frac{f_{2k}}{f_{(2n+1)k}^2 - f_k^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{2nk+1}}{f_{2nk}} - \frac{f_{(2n+2)k+1}}{f_{(2n+2)k}} \right] \\ &= \frac{f_{2k+1}}{f_{2k}} - \alpha. \end{aligned} \tag{3.5}$$

On the other hand, suppose $g_n = l_n$. Lemma 2.2, coupled with identities (1.1) and (1.2), yields

$$\begin{aligned} \frac{\Delta^2 f_{2k}}{l_{(2n+1)k}^2 + \Delta^2 f_k^2} &= \frac{l_{(2n+2)k+1} l_{2nk} - l_{(2n+2)k} l_{2nk+1}}{l_{(2n+2)k} l_{2nk}}; \\ \sum_{n=1}^{\infty} \frac{-\Delta^2 f_{2k}}{l_{(2n+1)k}^2 + \Delta^2 f_k^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{2nk+1}}{l_{2nk}} - \frac{l_{(2n+2)k+1}}{l_{(2n+2)k}} \right] \\ &= \frac{l_{2k+1}}{l_{2k}} - \alpha. \end{aligned} \tag{3.6}$$

Combining the two cases, we get the desired result. □

In particular, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 - 1} &= \frac{3}{2} - \frac{\sqrt{5}}{2}; & \sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 + 5} &= -\frac{1}{6} + \frac{\sqrt{5}}{10}; \\ \sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 1} &= \frac{7}{18} - \frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 5} &= -\frac{1}{14} + \frac{\sqrt{5}}{30}; \\ \sum_{n=1}^{\infty} \frac{1}{F_{3(2n+1)}^2 - 4} &= \frac{9}{64} - \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L_{3(2n+1)}^2 + 20} &= -\frac{1}{36} + \frac{\sqrt{5}}{80}. \end{aligned}$$

The following theorem is an application of Lemma 2.3.

Theorem 3.3. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu^* f_{4k}}{g_{(4n-1)k}^2 - (-1)^k \mu\nu^* f_{2k}^2} = \frac{g_{k+1}}{g_k} - \alpha. \tag{3.7}$$

Proof. Let $g_n = f_n$. Lemma 2.3, coupled with identities (1.1) and (1.2), yields

$$\begin{aligned} \frac{(-1)^k f_{4k}}{f_{(4n-1)k}^2 - (-1)^k f_{2k}^2} &= \frac{f_{(4n+1)k} f_{(4n-3)k+1} - f_{(4n+1)k+1} f_{(4n-3)k}}{f_{(4n+1)k} f_{(4n-3)k}}; \\ \sum_{n=1}^{\infty} \frac{(-1)^k f_{4k}}{f_{(4n-1)k}^2 - (-1)^k f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(4n-3)k+1}}{f_{(4n-3)k}} - \frac{f_{(4n+1)k+1}}{f_{(4n+1)k}} \right] \\ &= \frac{f_{k+1}}{f_k} - \alpha. \end{aligned} \tag{3.8}$$

On the other hand, let $g_n = l_n$. With identities (1.1) and (1.2), Lemma 2.3 yields

$$\begin{aligned}
\frac{-(-1)^k \Delta^2 f_{4k}}{l^2_{(4n-1)k} + (-1)^k \Delta^2 f^2_{2k}} &= \frac{l_{(4n+1)k} l_{(4n-3)k+1} - l_{(4n+1)k+1} l_{(4n-3)k}}{l_{(4n+1)k} l_{(4n-3)k}};\\
\sum_{n=1}^{\infty} \frac{-(-1)^k \Delta^2 f_{4k}}{l^2_{(4n-1)k} + (-1)^k \Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty} \left[\frac{l_{(4n-3)k+1}}{l_{(4n-3)k}} - \frac{l_{(4n+1)k+1}}{l_{(4n+1)k}} \right]\\
&= \frac{l_{k+1}}{l_k} - \alpha. \qquad (3.9)
\end{aligned}$$

By combining the two cases, we get the desired result. □

It follows from Theorem 3.3 that

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{4n-1} + 1} = -\frac{1}{6} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{4n-1} - 5} = \frac{1}{6} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{2(4n-1)} - 9} = \frac{1}{14} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{2(4n-1)} + 45} = -\frac{1}{126} + \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{3(4n-1)} + 64} = -\frac{1}{144} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{3(4n-1)} - 320} = \frac{1}{576} - \frac{\sqrt{5}}{1,440}.$$

Theorem 3.4. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\nu^* f_{4k}}{g^2_{4nk} - \mu\nu^* f^2_{2k}} = \frac{g_{2k+1}}{g_{2k}} - \alpha. \qquad (3.10)$$

Proof. Suppose $g_n = f_n$. Lemma 2.4, coupled with identities (1.1) and (1.2), yields

$$\begin{aligned}
\frac{f_{4k}}{f^2_{4nk} - f^2_{2k}} &= \frac{f_{(4n+2)k} f_{(4n-2)k+1} - f_{(4n+2)k+1} f_{(4n-2)k}}{f_{(4n+2)k} f_{(4n-2)k}};\\
\sum_{n=1}^{\infty} \frac{f_{4k}}{f^2_{4nk} - f^2_{2k}} &= \sum_{n=1}^{\infty} \left[\frac{f_{(4n-2)k+1}}{f_{(4n-2)k}} - \frac{f_{(4n+2)k+1}}{f_{(4n+2)k}} \right]\\
&= \frac{f_{2k+1}}{f_{2k}} - \alpha. \qquad (3.11)
\end{aligned}$$

On the other hand, let $g_n = l_n$. It then follows by identities (1.1) and (1.2), and Lemma 2.4 that

$$\begin{aligned}
\frac{-\Delta^2 f_{4k}}{l^2_{4nk} + \Delta^2 f^2_{2k}} &= \frac{l_{(4n+2)k} l_{(4n-2)k+1} - l_{(4n+2)k+1} l_{(4n-2)k}}{l_{(4n+2)k} l_{(4n-2)k}};\\
\sum_{n=1}^{\infty} \frac{-\Delta^2 f_{4k}}{l^2_{4nk} + \Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty} \left[\frac{l_{(4n-2)k+1}}{l_{(4n-2)k}} - \frac{l_{(4n+2)k+1}}{l_{(4n+2)k}} \right]\\
&= \frac{l_{2k+1}}{l_{2k}} - \alpha. \qquad (3.12)
\end{aligned}$$

Combining the two cases yields the desired result. □

It follows by the theorem that

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{8n}^2 - 9} = \frac{1}{18} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n}^2 + 45} = -\frac{1}{98} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440}.$$

Next, we present an application of Lemma 2.5.

Theorem 3.5. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu^* f_{4k}}{g_{(4n+1)k}^2 - (-1)^k \mu\nu^* f_{2k}^2} = \frac{g_{3k+1}}{g_{3k}} - \alpha. \tag{3.13}$$

Proof. Suppose $g_n = f_n$. With identities (1.1) and (1.2), Lemma 2.5 yields

$$\frac{(-1)^k f_{4k}}{f_{(4n+1)k}^2 - (-1)^k f_{2k}^2} = \frac{f_{(4n+3)k} f_{(4n-1)k+1} - f_{(4n+3)k+1} f_{(4n-1)k}}{f_{(4n+3)k} f_{(4n-1)k}};$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^k f_{4k}}{f_{(4n+1)k}^2 - (-1)^k f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(4n-1)k+1}}{f_{(4n-1)k}} - \frac{f_{(4n+3)k+1}}{f_{(4n+3)k}}\right] \\ &= \frac{f_{3k+1}}{f_{3k}} - \alpha. \end{aligned} \tag{3.14}$$

On the other hand, suppose $g_n = l_n$. Using identities (1.2) and (2.1), and Lemma 2.5, we get

$$\frac{(-1)^{k+1} \Delta^2 f_{4k}}{l_{(4n+1)k}^2 + (-1)^k \Delta^2 f_{2k}^2} = \frac{l_{(4n+3)k} l_{(4n-1)k+1} - l_{(4n+3)k+1} l_{(4n-1)k}}{l_{(4n+3)k} l_{(4n-1)k}};$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{k+1} \Delta^2 f_{4k}}{l_{(4n+1)k}^2 + (-1)^k \Delta^2 f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(4n-1)k+1}}{l_{(4n-1)k}} - \frac{l_{(4n+3)k+1}}{l_{(4n+3)k}}\right] \\ &= \frac{l_{3k+1}}{l_{3k}} - \alpha. \end{aligned} \tag{3.15}$$

By combining equations (3.14) and (3.15), we get the desired result. □

It follows from Theorem 3.5 that

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{12} - \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(4n+1)}^2 - 9} = \frac{3}{56} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+1)}^2 + 45} = -\frac{2}{189} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(4n+1)}^2 + 64} = -\frac{19}{2,448} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(4n+1)}^2 - 320} = \frac{17}{10,944} - \frac{\sqrt{5}}{1,440}.$$

Finally, we showcase an application of Lemma 2.6.

Theorem 3.6. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\nu^* f_{4k}}{g_{(4n+2)k}^2 - \mu\nu^* f_{2k}^2} = \frac{g_{4k+1}}{g_{4k}} - \alpha. \tag{3.16}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.2), and Lemma 2.6, we get

$$\begin{aligned}
\frac{f_{4k}}{f^2_{(4n+2)k} - f^2_{2k}} &= \frac{f_{(4n+4)k}f_{4nk+1} - f_{(4n+4)k+1}f_{4nk}}{f_{(4n+4)k}f_{4nk}};\\
\sum_{n=1}^{\infty} \frac{f_{4k}}{f^2_{(4n+2)k} - f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{f_{4nk+1}}{f_{4nk}} - \frac{f_{(4n+4)k+1}}{f_{(4n+4)k}}\right]\\
&= \frac{f_{4k+1}}{f_{4k}} - \alpha. \qquad (3.17)
\end{aligned}$$

On the other hand, let $g_n = l_n$. With identities (1.1) and (1.2), Lemma 2.6 yields

$$\begin{aligned}
\frac{-\Delta^2 f_{4k}}{l^2_{(4n+2)k} + \Delta^2 f^2_{2k}} &= \frac{l_{(4n+4)k}l_{4nk+1} - l_{(4n+4)k+1}l_{4nk}}{l_{(4n+4)k}l_{4nk}};\\
\sum_{n=1}^{\infty} \frac{-\Delta^2 f_{4k}}{l^2_{(4n+2)k} + \Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{l_{4nk+1}}{l_{4nk}} - \frac{l_{(4n+4)k+1}}{l_{(4n+4)k}}\right]\\
&= \frac{l_{4k+1}}{l_{4k}} - \alpha. \qquad (3.18)
\end{aligned}$$

By combining equations (3.17) and (3.18), we get the desired result. □

This theorem yields

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{F^2_{4n+2} - 1} &= \frac{7}{18} - \frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{4n+2} + 5} &= -\frac{1}{14} + \frac{\sqrt{5}}{30};\\
\sum_{n=1}^{\infty}\frac{1}{F^2_{2(4n+2)} - 9} &= \frac{47}{882} - \frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{2(4n+2)} + 45} &= -\frac{1}{94} + \frac{\sqrt{5}}{210};\\
\sum_{n=1}^{\infty}\frac{1}{F^2_{3(4n+2)} - 64} &= \frac{161}{20,736} - \frac{\sqrt{5}}{288}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{3(4n+2)} + 320} &= -\frac{1}{644} + \frac{\sqrt{5}}{1,440}.
\end{aligned}$$

3.1. **Gibonacci Delights.** We can extract interesting dividends from the theorems. Using Theorems 3.1 and 3.2, we get

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{1}{F^2_{2n} - 1} &= \sum_{n=1}^{\infty}\frac{1}{F^2_{2(2n)} - 1} + \sum_{n=1}^{\infty}\frac{1}{F^2_{2(2n+1)} - 1}\\
&= \frac{8}{9} - \frac{\sqrt{5}}{3};\\
\sum_{n=2}^{\infty}\frac{1}{L^2_{2n} + 5} &= \sum_{n=1}^{\infty}\frac{1}{L^2_{2(2n)} + 5} + \sum_{n=1}^{\infty}\frac{1}{L^2_{2(2n+1)} + 5}\\
&= -\frac{8}{63} + \frac{\sqrt{5}}{15}.
\end{aligned}$$

Theorems 3.3 and 3.5 yield

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 + 1} &= \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n-1}^2 + 1} \qquad (3.19)\\
&= -\frac{1}{2} + \frac{\sqrt{5}}{3};\\
\sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(4n+1)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{2(4n-1)}^2 - 9}\\
&= \frac{1}{8} - \frac{\sqrt{5}}{21}; \qquad (3.20)\\
\sum_{n=1}^{\infty} \frac{1}{F_{3(2n+1)}^2 + 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{3(4n+1)}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{2(4n-1)}^2 + 64}\\
&= -\frac{1}{68} + \frac{\sqrt{5}}{144};\\
\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 5} &= \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n-1}^2 - 5}\\
&= \frac{1}{4} - \frac{\sqrt{5}}{15};\\
\sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+1)}^2 + 45} + \sum_{n=1}^{\infty} \frac{1}{L_{2(4n-1)}^2 + 45}\\
&= -\frac{1}{54} + \frac{\sqrt{5}}{105}; \qquad (3.21)\\
\sum_{n=1}^{\infty} \frac{1}{L_{3(2n+1)}^2 - 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{3(4n+1)}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{3(4n-1)}^2 - 320}\\
&= \frac{1}{304} - \frac{\sqrt{5}}{720}.
\end{aligned}$$

It follows by Theorems 3.4 and 3.6 that

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{4(2n+1)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{4(2n)}^2 - 9}\\
&= \frac{16}{147} - \frac{\sqrt{5}}{21}; \qquad (3.22)\\
\sum_{n=2}^{\infty} \frac{1}{F_{6n}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{6(2n+1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{6(2n)}^2 - 64}\\
&= \frac{323}{20,736} - \frac{\sqrt{5}}{144};\\
\sum_{n=2}^{\infty} \frac{1}{L_{4n}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{4(2n)}^2 + 45} + \sum_{n=1}^{\infty} \frac{1}{L_{4(2n+1)}^2 + 45}
\end{aligned}$$

$$= -\frac{48}{2,303} + \frac{\sqrt{5}}{105}. \qquad (3.23)$$

3.2. **Additional Delighters.** It follows by equations (3.20) and (3.22) that

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{F_{2n}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{2(2n)}^2 - 9} \\ &= \frac{275}{1,176} - \frac{2\sqrt{5}}{21}. \end{aligned}$$

Likewise, equations (3.21) and (3.23) yield

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{L_{2n}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 45} + \sum_{n=2}^{\infty} \frac{1}{L_{2(2n)}^2 + 45} \\ &= -\frac{4,895}{124,362} + \frac{2\sqrt{5}}{105}. \end{aligned}$$

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