ADDITIONAL SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES REVISITED

THOMAS KOSHY

ABSTRACT. We explore two infinite sums involving gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2+4}$, $2\alpha = x + \Delta$, and $2\beta = x - \Delta$.

It follows by the Binet-like formulas that $\lim\limits_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$ and $\lim\limits_{m\to\infty} \dfrac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Fundamental Gibonacci Identities. Gibonacci polynomials satisfy the following properties [2, 3, 4]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} -(-1)^{n+k} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1.1}$$

$$g_{n+k+1}g_{n-k} - g_{n+k}g_{n-k+1} = \begin{cases} -(-1)^{n+k} f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_{2k}, & \text{otherwise;} \end{cases} \tag{1.2}$$

$$g_{n+k+1}g_{n-k} + g_{n+k}g_{n-k+1} = \begin{cases} \dfrac{1}{\Delta^2}[2l_{2n+1} - (-1)^{n+k} x l_{2k}], & \text{if } g_n = f_n; \\ 2l_{2n+1} + (-1)^{n+k} x l_{2k}, & \text{otherwise.} \end{cases} \tag{1.3}$$

These properties can be confirmed using the Binet-like formulas. Identity (1.2) is a gibonacci polynomial extension of *d'Ocagne identity* [2].

It follows by identities (1.2) and (1.3) that

$$g_{n+k+1}^2 g_{n-k}^2 - g_{n+k}^2 g_{n-k+1}^2 = \begin{cases} -\dfrac{(-1)^{n+k}}{\Delta^2}[2l_{2n+1} - (-1)^{n+k} x l_{2k}] f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 [2l_{2n+1} + (-1)^{n+k} x l_{2k}] f_{2k}, & \text{otherwise.} \end{cases} \tag{1.4}$$

2. A Telescoping Gibonacci Sum

Using recursion, we will now explore a telescoping gibonacci sum.

Lemma 2.1. *Let k and λ are positive integers. Then*

$$\sum_{n=1}^{\infty}\left[\frac{g_{2nk+2}^{\lambda}}{g_{2nk+1}^{\lambda}}-\frac{g_{2(n+1)k+2}^{\lambda}}{g_{2(n+1)k+1}^{\lambda}}\right]=\frac{g_{2k+2}^{\lambda}}{g_{2k+1}^{\lambda}}-\alpha^{\lambda}. \tag{2.1}$$

Proof. Using recursion [2], we will first confirm that

$$\sum_{n=1}^{m}\left[\frac{g_{2nk+2}^{\lambda}}{g_{2nk+1}^{\lambda}}-\frac{g_{2(n+1k+2}^{\lambda}}{g_{2(n+1)k+1}^{\lambda}}\right]=\frac{g_{2k+2}^{\lambda}}{g_{2k+1}^{\lambda}}-\frac{g_{2(m+1)k+2}^{\lambda}}{g_{2(m+1)k+1}^{\lambda}}. \tag{2.2}$$

To this end, we let A_m denote the left-hand side of this equation and B_m its right-hand side. Then

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

With recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

This confirms the validity of equation (2.2).

Because $\lim\limits_{m\to\infty}\dfrac{g_{m+1}}{g_m}=\alpha$, equation (2.2) yields the desired result. □

3. Gibonacci Sums

The above lemma, coupled with identities (1.1), (1.2), and (1.4), plays a major role in our explorations. In the interest of brevity, we now let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \nu = \begin{cases} -1, & \text{if } g_n = f_n; \\ 1, & \text{otherwise;} \end{cases}$$

$$\mu^* = \begin{cases} \dfrac{1}{\Delta^2}, & \text{if } g_n = f_n; \\ 1, & \text{otherwise.} \end{cases}$$

The next result invokes Lemma 2.1 twice with $\lambda = 1$.

Theorem 3.1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{\mu\nu f_{2k}}{g_{(2n+1)k+1}^2-\mu\nu g_k^2}=\frac{g_{2k+2}}{g_{2k+1}}-\alpha. \tag{3.1}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.2), Lemma 2.1 yields

$$\begin{aligned} \frac{f_{2k}}{f_{(2n+1)k+1}^2+f_k^2} &= \frac{f_{2(n+1)k+2}f_{2nk+1}-f_{2(n+1)k+1}f_{2nk+2}}{f_{(2n+1)k+1}f_{2nk+1}}; \\ \sum_{n=1}^{\infty}\frac{-f_{2k}}{f_{(2n+1)k+1}^2+f_k^2} &= \sum_{n=1}^{\infty}\left[\frac{f_{2nk+2}}{f_{2nk+1}}-\frac{f_{2(n+1)k+2}}{f_{2(n+1)k+1}}\right] \\ &= \frac{f_{2k+2}}{f_{2k+1}}-\alpha. \end{aligned}$$

On the other hand, let $g_n = l_n$. It then follows by the two aforementioned identities and Lemma 2.1 that

$$\begin{aligned}
\frac{\Delta^2 f_{2k}}{l^2_{(2n+1)k+1} - \Delta^2 f_k^2} &= \frac{l_{2(n+1)k+1} l_{2nk+2} - l_{2(n+1)k+2} l_{2nk+1}}{l_{(2n+1)k+1} l_{2nk+1}}; \\
\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k}}{l^2_{2nk+k+1} - \Delta^2 f_k^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{2nk+2}}{l_{2nk+1}} - \frac{l_{2(n+1)k+2}}{l_{2(n+1)k+1}} \right] \\
&= \frac{l_{2k+2}}{l_{2k+1}} - \alpha.
\end{aligned}$$

By combining the two cases, we get the desired result. □

Using the identity $F^2_{n+1} + F^2_n = F_{2n+1}$ [2, p. 57], it follows from equation (3.1) that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{F_{2k}}{F^2_{(2n+1)k+1} + F_k^2} &= \frac{F_{2k}}{F^2_{k+1} + F_k^2} - \frac{F_{2k+2}}{F_{2k+1}} + \alpha \\
&= \frac{F_{2k}}{F_{2k+1}} - \frac{F_{2k+2}}{F_{2k+1}} + \alpha \\
&= -\beta.
\end{aligned}$$

Likewise, with the identities $L^2_{n+1} - 5F^2_n = L_{2n+1}$ [2, p. 59] and $L_{n+2} + 5F_n = 3L_{n+1}$, equation (3.1) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{5F_{2k}}{L^2_{(2n+1)k+1} - 5F_k^2} &= \frac{5F_{2k}}{L^2_k - 5F^2_k} + \frac{L_{2k+2}}{L_{2k+1}} - \alpha \\
&= \frac{5F_{2k}}{L_{2k+1}} + \frac{L_{2k+2}}{L_{2k+1}} - \alpha \\
&= \frac{L_{2k+2} + 5F_{2k}}{L_{2k+1}} - \alpha \\
&= 2 + \beta.
\end{aligned}$$

For clarity, we combine the two formulas into one line:

$$\sum_{n=0}^{\infty} \frac{F_{2k}}{F^2_{(2n+1)k+1} + F_k^2} = -\beta; \qquad \sum_{n=0}^{\infty} \frac{5F_{2k}}{L^2_{(2n+1)k+1} - 5F_k^2} = 2 + \beta.$$

In particular, we then get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{F^2_{2n+2} + 1} &= -\frac{1}{2} + \frac{\sqrt{5}}{2} \text{ [3]}; & \sum_{n=0}^{\infty} \frac{1}{L^2_{2n+2} - 5} &= \frac{1}{2} - \frac{\sqrt{5}}{10}; \\
\sum_{n=0}^{\infty} \frac{1}{F^2_{4n+3} + 1} &= -\frac{1}{6} + \frac{\sqrt{5}}{6}; & \sum_{n=0}^{\infty} \frac{1}{L^2_{4n+3} - 5} &= \frac{1}{6} - \frac{\sqrt{5}}{30}; \\
\sum_{n=0}^{\infty} \frac{1}{F^2_{6n+4} + 4} &= -\frac{1}{16} + \frac{\sqrt{5}}{16}; & \sum_{n=0}^{\infty} \frac{1}{L^2_{6n+4} - 20} &= \frac{1}{16} - \frac{\sqrt{5}}{80}; \\
\sum_{n=0}^{\infty} \frac{1}{F^2_{8n+5} + 9} &= -\frac{1}{42} + \frac{\sqrt{5}}{42}; & \sum_{n=0}^{\infty} \frac{1}{L^2_{8n+5} - 45} &= \frac{1}{42} - \frac{\sqrt{5}}{210}.
\end{aligned}$$

The next theorem employs Lemma 2.1 with $\lambda = 2$.

Theorem 3.2. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\nu[2l_{2(2n+1)k+3} - \nu x l_{2k}]f_{2k}}{\mu^*[g^2_{(2n+1)k+1} - \mu\nu f_k^2]^2} = \frac{g^2_{2k+2}}{g^2_{2k+1}} - \alpha^2. \tag{3.2}$$

Proof. Let $g_n = f_n$. With identities (1.1) and (1.4), Lemma 2.1 yields

$$\begin{aligned} \frac{[2l_{2(2n+1)k+3} + x l_{2k}]f_{2k}}{\Delta^2[f^2_{(2n+1)k+1} + f_k^2]^2} &= \frac{f^2_{2(n+1)k+2}f^2_{2nk+1} - f^2_{2(n+1)k+1}f^2_{2nk+2}}{f^2_{(2n+1)k+1}f^2_{2nk+1}}; \\ \sum_{n=1}^{\infty} \frac{-[2l_{2(2n+1)k+3} + x l_{2k}]f_{2k}}{\Delta^2[f^2_{(2n+1)k+1} + f_k^2]^2} &= \sum_{n=1}^{\infty} \left[\frac{f^2_{2nk+2}}{f^2_{2nk+1}} - \frac{f^2_{2(n+1)k+2}}{f^2_{2(n+1)k+1}}\right] \\ &= \frac{f^2_{2k+2}}{f^2_{2k+1}} - \alpha^2. \end{aligned}$$

On the other hand, let $g_n = l_n$. It then follows by the same identities and Lemma 2.1 that

$$\begin{aligned} \frac{\Delta^2[2l_{2(2n+1)k+3} - x l_{2k}]f_{2k}}{[l^2_{(2n+1)k+1} - \Delta^2 f_k^2]^2} &= \frac{l^2_{2(n+1)k+1}l^2_{2nk+2} - l^2_{2(n+1)k+2}l^2_{2nk+1}}{l^2_{2(n+1)k+1}l^2_{2nk+1}}; \\ \sum_{n=1}^{\infty} \frac{\Delta^2[2l_{2(2n+1)k+3} - x l_{2k}]f_{2k}}{[l^2_{(2n+1)k+1} - \Delta^2 f_k^2]^2} &= \sum_{n=1}^{\infty} \left[\frac{l^2_{2nk+2}}{l^2_{2nk+1}} - \frac{l^2_{2(n+1)k+2}}{l^2_{2(n+1)k+1}}\right] \\ &= \frac{l^2_{2k+2}}{l^2_{2k+1}} - \alpha^2, \end{aligned}$$

By merging the two cases, we get the desired result. □

This theorem yields

$$\sum_{n=1}^{\infty} -\frac{(2L_{4nk+2k+3} + L_{2k})F_{2k}}{5(F^2_{2nk+k+1} + F_k^2)^2} = \frac{F^2_{2k+2}}{F^2_{2k+1}} - \alpha^2; \qquad \sum_{n=1}^{\infty} \frac{5(2L_{4nk+2k+3} - L_{2k})F_{2k}}{(L^2_{2nk+k+1} - 5F_k^2)^2} = \frac{L^2_{2k+2}}{L^2_{2k+1}} - \alpha^2.$$

It then follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2L_{4n+5}+3}{(F^2_{2n+2}+1)^2} &= -\frac{15}{4} + \frac{5\sqrt{5}}{2}; & \sum_{n=1}^{\infty} \frac{2L_{4n+5}-3}{(L^2_{2n+2}-5)^2} &= \frac{5}{16} - \frac{\sqrt{5}}{10}; \\ \sum_{n=1}^{\infty} \frac{2L_{8n+7}+7}{(F^2_{4n+3}+1)^2} &= -\frac{53}{30} + \frac{5\sqrt{5}}{6}; & \sum_{n=1}^{\infty} \frac{2L_{8n+7}-7}{(L^2_{4n+3}-5)^2} &= \frac{19}{242} - \frac{\sqrt{5}}{30}; \\ \sum_{n=1}^{\infty} \frac{2L_{12n+9}+18}{(F^2_{6n+4}+4)^2} &= -\frac{1,875}{2,704} + \frac{5\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{2L_{12n+9}-18}{(L^2_{6n+4}-20)^2} &= \frac{379}{13,456} - \frac{\sqrt{5}}{80}; \\ \sum_{n=1}^{\infty} \frac{2L_{16n+11}+47}{(F^2_{8n+5}+9)^2} &= -\frac{6,455}{24,276} + \frac{5\sqrt{5}}{42}; & \sum_{n=1}^{\infty} \frac{2L_{16n+11}-47}{(L^2_{8n+5}-45)^2} &= \frac{431}{40,432} - \frac{\sqrt{5}}{210}. \end{aligned}$$

4. Acknowledgment

The author thanks the reviewer for a careful reading of the article, and for their encouraging words and constructive suggestions.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[3] T. Koshy, *Sums involving two classes of gibonacci polynomials*, The Fibonacci Quarterly, **61.4** (2023), 312–320.
[4] T. Koshy, *Sums involving two classes of gibonacci polynomials revisited*, The Fibonacci Quarterly, **61.4** (2023), 327–333.

MSC2020: Primary 11B37, 11B39, 11C08.

Department of Mathematics, Framingham State University, Framingham, MA 01701, USA
Email address: tkoshy@emeriti.framingham.edu