DELETING TERMS OF THE DIVERGENT p-SERIES AND RECIPROCALS OF PRIMES SERIES USING THE THUE-MORSE SEQUENCE

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Abstract. By replacing the numerator in the nth term of the divergent p-series and the reciprocals of primes series with the nth term of the Thue-Morse sequence, one can produce a deletion of terms in the said series, which we show remains divergent. A connection is also revealed, between the sequence $(a_n)_{n\geq 1}$ defined as the largest power of two to divide an integer n and the ordinary generating function for the Thue-Morse sequence. In addition, we provide a new elementary proof that the sequence $(a_n)_{n\geq 1}$ is square free in the context of combinatorics on words.

1. Introduction

The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is known to possess many intriguing properties. As an example, given the divergence of the harmonic series, it is known that the partial sums $S_N=\sum_{n=1}^{N}\frac{1}{n}$ grow unboundedly in such a way that S_N is never an integer for $N>1$. However, it is when one is addressing the process of deleting terms that the truly counter-intuitive nature of the harmonic series becomes apparent. Take for example the famous result of Euler, who proved that if we delete all the reciprocals of the composite numbers, including unity, then the resulting series, $\sum_{n=1}^{\infty}\frac{1}{p_n}$, where p_n is the nth prime number, remains divergent. Alternatively, consider the sequence of perfect powers without repetition, in *The On-Line Encyclopedia of Integer Sequences* (or OEIS) [8, A001597]. In particular, that sequence has terms of the form m^n with $m,\, n\in\mathbb{N}\backslash\{1\}$, where for example $16=2^4=4^2$ would appear exactly once in the sequence. Denote by P the infinite set whose elements consist only of the terms of the sequence of perfect powers without repetition, that is $\mathrm{P}=\{4,8,9,16,25,27,32,\ldots\}$. Then it can be shown [4, p. 66], that by deleting all terms of the harmonic series, not of the form $\frac{1}{n-1}$, where $n\in\mathrm{P}$, yields a convergent series and moreover, has a sum

$$\sum_{n\in\mathrm{P}}\frac{1}{n-1}=1.$$

In this paper, we shall achieve a deletion of terms of the divergent p-series, $\sum_{n=1}^{\infty}\frac{1}{n^p}$, where $0<p\leq 1$, by replacing the numerator of the nth term with the nth term of the Thue-Morse sequence [8, A010060]. Denoting the Thue-Morse sequence, that contains only 0s and 1s, and its corresponding ordinary generating function by $(t_n)_{n\geq 0}$ and $T(x)=\sum_{n=0}^{\infty}t_nx^n$, respectively, our aim here will be to show that the resulting deleted p-series, namely $\sum_{n=1}^{\infty}\frac{t_n}{n^p}$ remains divergent for $0<p\leq 1$. As will be seen, the divergence of this deleted p-series will be arrived at by exploiting a well-known functional equation in (2.1) for $T(x)$. In addition, as a consequence of the prime number theorem, we shall also show that the deleted series of reciprocals of primes, namely $\sum_{n=1}^{\infty}\frac{t_n}{p_n}$, also remains divergent. Continuing with the theme of deleting the terms of a series, we will consider the convergent geometric series with all terms deleted that are not of the form x^{2^n} for $n=0,1,\ldots$. Using again the functional equation for

$T(x)$, one can relate the sum of the resulting deleted series to $T(x)$ and a double infinite series involving the terms of the Thue-Morse sequence via the infinite series identity as follows:

$$\sum_{r=0}^{\infty} x^{2^r} = \frac{x}{1-x} - x^2T(x) - \sum_{n=1}^{\infty}\sum_{i=0}^{\infty} t_i(x^{2^n(i+2)} + x^{2^{n-1}(2i+3)} - x^{2^n(i+1)}).$$

In addition, consider now the sequence $(a_n)_{n\geq 1}$ defined as the largest power of two to divide n, that is in the OEIS [8, A001511]. Again, as a consequence of the functional equation for $T(x)$, we shall uncover a connection between $T(x)$ and the sequence $(a_n)_{n\geq 1}$ via another infinite series identity. In the final section of the paper, a new elementary proof of a result of Guay-Paquet and Shallit [5] will be given. Their result states that the sequence $(a_n)_{n\geq 1}$ is square free, and that it does not contain two consecutive identical segments.

2. Deleted p-Series and Reciprocals of Primes Series

It is relatively simple matter to show that a binary sequence of length four or more contains a square, that is, contains two consecutive identical segments. In view of this result, Thue [9] and Morse [7] independently proposed the construction of an infinite binary sequence that was free from cubes. That is, it does not contain three consecutive identical segments. The initial fragment of the resulting cube free sequence, known today as the Thue-Morse sequence, is given as follows:

$$(t_n) = (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, \ldots).$$

The Thue-Morse sequence has many nontrivial properties that have found numerous applications in mathematics. There are also many equivalent ways to define the Thue-Morse sequence, however we state the most commonly employed as follows.

Definition 2.1. *If t_n is the nth term in the Thue-Morse sequence, then t_n is equal to the number of ones in the base 2 expansion of $n \pmod 2$.*

The following result has appeared in numerous papers; interested readers can consult [1] for a proof of the functional equation in (2.1) that the ordinary generating function $T(x)$ satisfies.

Lemma 2.2. *Suppose $T(x)$ denotes the ordinary generating function of the Thue-Morse sequence $(t_n)_{n\geq 0}$, that is, $T(x) = \sum_{n=0}^{\infty} t_n x^n$ defined for $|x| < 1$. Then, $T(x)$ satisfies the functional equation as follows:*

$$T(x) - (1-x)T(x^2) = \frac{x}{1-x^2}. \tag{2.1}$$

By applying the functional equation in (2.1) for $T(x)$, we can now establish the divergence of the family of deleted p-series obtained using the Thue-Morse sequence. We should note that the related family of the Thue-Morse Dirichlet series, defined as $\sum_{n=1}^{\infty} \frac{\varepsilon_{n-1}}{n^s}$, where $\varepsilon_n = (-1)^{t_n}$ and s a fixed complex number, has also been studied [2]. Moreover the convergence of such a series for $s = 1$ has recently been established [3]. One can interpret the following result as stating that for a fixed $0 < p \leq 1$, the p-series of positive integers having an exclusively odd Hamming weight, when expressed in base 2, is divergent.

Theorem 2.3. *For a fixed $0 < p \leq 1$, the deleted p-series using the Thue-Morse sequence, that is, the series of the form*

$$\sum_{n=1}^{\infty} \frac{t_n}{n^p},$$

remains divergent.

Proof. For a fixed $0 < p \leq 1$, we have $\frac{t_n}{n} \leq \frac{t_n}{n^p}$ for all $n = 1, 2, 3, \ldots$. Thus, it suffices to establish the divergence of the series $\sum_{n=1}^{\infty} \frac{t_n}{n}$. Recalling that the radius of convergence of $T(x)$ is one, consider for a fixed $0 < r < 1$, the subset $[0, r]$ of the interval of convergence of $T(x)$. Clearly $(1-x)T(x^2) > 0$ for all $x \in [0, r]$, and so from the functional equation in (2.1), we deduce that $T(x) > \frac{x}{1-x^2}$ for all $x \in [0, r]$. Thus, as $t_0 = 0$, observe, after an interchange of integration and summation, that

$$\sum_{n=1}^{\infty} t_n \frac{r^{n+1}}{n+1} = \int_0^r T(x)\, dx > \int_0^r \frac{x}{1-x^2}\, dx = \ln\left(\frac{1}{\sqrt{1-r^2}}\right).$$

However, because $\frac{t_n}{n} \geq t_n \frac{r^{n+1}}{n+1}$, one can conclude, for any fixed $0 < r < 1$, that

$$\sum_{n=1}^{\infty} \frac{t_n}{n} > \ln\left(\frac{1}{\sqrt{1-r^2}}\right),$$

from which the result now readily follows upon taking the limit as $r \to 1^-$. □

If we define the complement of the Thue-Morse sequence as $\overline{t_n} = 1 - t_n$ for $n \geq 1$, it is straightforward to show that the corresponding generating function $\overline{T}(x) = \sum_{n=1}^{\infty} \overline{t_n} x^n$ satisfies the functional equation

$$\overline{T}(x) - (1-x)\overline{T}(x^2) = \frac{x^3}{1-x^2}.$$

Thus, it is immediate, from the proof of Theorem 2.3, that for a fixed $0 < p \leq 1$, the p-series of positive integers, having an exclusively even Hamming weight when expressed in base 2, is also divergent. An analogous result can now be shown to hold for the divergent series of reciprocals of primes by an application of the prime number theorem as follows.

Theorem 2.4. *The deleted series of the reciprocals of primes using the Thue-Morse sequence, that is, the series of the form*

$$\sum_{n=1}^{\infty} \frac{t_n}{p_n},$$

remains divergent.

Proof. We begin by noting that the interval $[1, \infty)$ on which the set of positive integers $n \geq 1$ are contained can be partitioned as follows:

$$[1, \infty) = \bigcup_{i=0}^{\infty} [2^i, 2^{i+1}).$$

If for a positive integer $m \geq 1$, we denote the $2^{m+1} - 1$ partial sums of the deleted series of reciprocals of primes by $S_{2^{m+1}-1}$, then

$$S_{2^{m+1}-1} = \sum_{i=0}^{m} \sum_{n \in [2^i, 2^{i+1})} \frac{t_n}{p_n} = \frac{1}{2} + \sum_{i=1}^{m} \sum_{n \in [2^i, 2^{i+1})} \frac{t_n}{p_n}. \tag{2.2}$$

Now for $i \geq 1$, every integer $n \in [2^i, 2^{i+1})$ has a base 2 representation consisting of a single 1, concatenated on the right with a finite sequence of 0s and 1s of length i. Consequently, a number $n \in [2^i, 2^{i+1})$ will have $t_n = 1$ if and only if the said finite sequence of length i has an even number of 1s. Thus, the number of $n \in [2^i, 2^{i+1})$ having $t_n = 1$ must be the finite sum of even indexed binomial coefficients, i.e., $\sum_{j\geq 0}\binom{i}{2j} = 2^{i-1}$, and so,

$$\sum_{n\in[2^i,2^{i+1})} \frac{t_n}{p_n} > \frac{2^{i-1}}{p_{2^{i+1}}}.$$

As a result, the partial sum in (2.2) for $m \geq 1$ must satisfy the inequality

$$S_{2^{m+1}-1} > \frac{1}{2} + \sum_{i=1}^{m} \frac{2^{i-1}}{p_{2^{i+1}}}. \tag{2.3}$$

Now via the prime number theorem, it is well known [6] that the nth prime $p_n \sim n\ln(n)$ as $n \to \infty$, and so,

$$\frac{2^{i-1}}{p_{2^{i+1}}} \sim \frac{2^{i-1}}{2^{i+1}\ln(2^{i+1})} = \frac{1}{(i+1)\ln(16)},$$

as $i \to \infty$. Clearly as $\sum_{i=1}^{\infty}\frac{1}{(i+1)\ln(16)}$ is a divergent series, we deduce, from the limit form of the comparison test for series, that $\sum_{i=1}^{\infty}\frac{2^{i-1}}{p_{2^{i+1}}}$ is also a divergent series of positive terms. Thus from (2.3), $S_{2^{m+1}-1} \to \infty$ as $m \to \infty$. □

As the sum of the even indexed binomial coefficients $\sum_{j\geq 0}\binom{i}{2j+1} = 2^{i-1}$, it is immediate, from the proof of Theorem 2.4, that the deleted series $\sum_{n=1}^{\infty}\frac{\overline{t_n}}{p_n}$ must also be divergent, where $\overline{t_n} = 1 - t_n$ is again the complement of the Thue-Morse sequence. We now derive an infinite series identity, relating the deleted geometric series $\sum_{r=0}^{\infty} x^{2^r}$ to the ordinary generating function $T(x)$.

Theorem 2.5. *If $T(x)$ is the ordinary generating function for the Thue-Morse sequence $(t_n)_{n\geq 0}$, then for $|x| < 1$,*

$$\sum_{r=0}^{\infty} x^{2^r} = \frac{x}{1-x} - x^2T(x) - \sum_{n=1}^{\infty}\sum_{i=0}^{\infty} t_i\left(x^{2^n(i+2)} + x^{2^{n-1}(2i+3)} - x^{2^n(i+1)}\right). \tag{2.4}$$

Proof. We first note for $|x| < 1$, $\sum_{n=1}^{\infty} x^{2n+1} = \frac{x^3}{1-x^2}$. By definition, the infinite series on the left side of (2.4) is obtained by deleting all terms of the form $x^{(2n+1)2^{k-1}}$, where $n = 1, 2, \ldots$ for each fixed $k = 1, 2, \ldots$ from the infinite series

$$x + x^2 + x^3 + x^4 + \cdots = \frac{x}{1-x}.$$

Consequently, we deduce that

$$\sum_{r=0}^{\infty} x^{2^r} = \frac{x}{1-x} - \sum_{k=1}^{\infty} \frac{x^{3\cdot 2^{k-1}}}{1-x^{2^k}},$$

however, by replacing x by $x^{2^{k-1}}$ in the functional equation for $T(x)$ in (2.1), we also find that

$$x^{2^k}\left(T(x^{2^{k-1}}) - (1 - x^{2^{k-1}})T(x^{2^k})\right) = \frac{x^{3\cdot 2^{k-1}}}{1-x^{2^k}}.$$

Thus,

$$\begin{aligned}\sum_{r=0}^{\infty} x^{2^r} &= \frac{x}{1-x} - \sum_{k=1}^{\infty} x^{2^k} T(x^{2^{k-1}}) - \sum_{k=1}^{\infty} x^{2^k}(1-x^{2^{k-1}})T(x^{2^k}) \\ &= \frac{x}{1-x} - x^2T(x) - \sum_{k=2}^{\infty} x^{2^k} T(x^{2^{k-1}}) - \sum_{k=1}^{\infty} x^{2^k}(1-x^{2^{k-1}})T(x^{2^k}) \\ &= \frac{x}{1-x} - x^2T(x) - \sum_{k=1}^{\infty} x^{2^{k+1}} T(x^{2^k}) - \sum_{k=1}^{\infty} x^{2^k}(1-x^{2^{k-1}})T(x^{2^k}) \\ &= \frac{x}{1-x} - x^2T(x) - \sum_{k=1}^{\infty} (x^{2^{k+1}} + x^{3\cdot 2^{k-1}} - x^{2^k})T(x^{2^k}). \qquad (2.5)\end{aligned}$$

The result now follows upon substituting the series $T(x^{2^n}) = \sum_{i=0}^{\infty} t_i x^{2^n i}$ into the right side of (2.5). ☐

By again applying the functional equation in (2.1), we can demonstrate a link between $T(x)$ and the sequence $(a_n)_{n\geq 1}$ defined as the largest power of two to divide n [8, A001511].

Theorem 2.6. *Let $(a_n)_{n\geq 1}$ denote the sequence defined as largest power of two to divide n. Then, the ordinary generating function for the Thue-Morse sequence $(t_n)_{n\geq 0}$ satisfies the infinite series identity as follows:*

$$T(x) = \sum_{n=1}^{\infty} \sum_{i=0}^{2^{a_n}-1} (-1)^{t_i} x^{i+n}.$$

Proof. Let $P_0(x) = 1$ and for integers $n = 1, 2 \ldots$, define the partial product $P_n(x) = \prod_{i=1}^{n}(1 - x^{2^{i-1}})$. After repeated application of the functional equation in (2.1), one finds that

$$T(x) = \sum_{n=0}^{N} \frac{P_n(x)x^{2^n}}{1 - x^{2^{n+1}}} + P_{N+1}(x)T(x^{2^{N+1}})$$

for all integers $N \geq 0$. Now for each fixed integer $N \geq 0$, observe that

$$\begin{aligned}0 \leq |T(x^{2^{N+1}})| \leq \sum_{n=1}^{\infty} |x|^{n2^{N+1}} &= |x|^{2^{N+1}} \sum_{n=1}^{\infty} |x|^{(n-1)2^{N+1}} \\ &\leq |x|^{2^{N+1}} \frac{1}{1-|x|},\end{aligned}$$

noting here the last inequality follows because for each fixed integer $N \geq 0$, the set $\{(n-1)2^{N+1} : n \in \mathbb{N}\} \subseteq \mathbb{N} \cup \{0\}$ and $|x| < 1$. Thus, $T(x^{2^{N+1}}) = o(1)$ as $N \to \infty$, and as $0 < P_{N+1}(x) < 2$ for $|x| < 1$, one concludes that

$$T(x) = \sum_{m=0}^{\infty} \frac{P_m(x)x^{2^m}}{1 - x^{2^{m+1}}}.$$

Expanding $\frac{1}{(1-x^{2^{m+1}})}$ as a geometric series, with common ratio $x^{2^{m+1}}$, we further see that

$$T(x) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} P_m(x)x^{2^m}x^{k2^{m+1}} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} P_m(x)x^{2^m(2k+1)}. \tag{2.6}$$

However, as the set $\{2^m(2k+1) : m, k \in \mathbb{N} \cup \{0\}\} = \mathbb{N}$, the double infinite series in (2.6) must be equal to $\sum_{n=1}^{\infty} P_{a_n} x^n$. The result now follows, after substituting $P_{a_n}(x) = \sum_{i=0}^{2^{a_n}-1}(-1)^{T_i}x^i$ into the previous infinite series. □

3. A New Proof the OEIS Sequence A001511 Is Square Free

As stated earlier, the Thue-Morse sequence was constructed to show the existence of an infinite binary sequence that was cube free; that is, it does not contain three consecutive identical blocks. The so-called *ruler sequence* $(a_n)_{n\geq1}$ defined as the largest power of two to divide an integer $n \geq 1$ [8, A001511] was shown to be similarly square free [5]. This was achieved by first concatenating all terms of the sequence $(a_n)_{n\geq1}$ to construct the following infinite word

$$\mathbf{w_2} = 01020103\cdots$$

over the alphabet of natural numbers $\mathbb{N}$. By applying a no-backtracking algorithm, the infinite word $\mathbf{w_2}$ was shown to be square free, thus, establishing that the sequence $(a_n)_{n\geq1}$ could never exhibit two consecutive segments that are identical. In this section, we shall provide an elementary number-theoretic proof that employs nothing more than the partitioning of the odd integers into residue classes modulo a fixed even integer to establish the square free property of the sequence $(a_n)_{n\geq1}$.

Theorem 3.1. *The OEIS sequence A001511 is square free, in the context of combinatorics on words.*

Proof. We begin with a simple observation, namely that for a segment of the sequence $(a_n)_{n\geq1}$ to be repeated by a consecutive segment, the initial left segment must necessarily have an even number of terms. In what follows, let N and E denote the number of terms and the smallest even integer, respectively within an arbitrary segment of the sequence $(a_n)_{n\geq1}$. In addition, let $E = 2^{c_1}q_1$ and $N = 2^{c_2}q_2$, where c_1 and c_2 are positive integers, and q_1 and q_2 are positive odd integers. To show the sequence $(a_n)_{n\geq1}$ is square free, we shall examine the largest power of two to divide an even number e in an arbitrary segment of $(a_n)_{n\geq1}$ having N terms, with the largest power of two to divide its translation $e + N$, in the following three cases:

Case 1 ($c_1 = c_2$). In this instance $a_E = c_1$, but because $E + N = 2^{c_1}(q_1 + q_2)$ and $q_1 + q_2$ is an even integer, we have that $a_{E+N} > a_E$. Thus, $a_E \neq a_{E+N}$.

Case 2 ($c_1 > c_2$). Again $a_E = c_1$, but because $E+N = 2^{c_2}(2^{c_1-c_2}q_1+q_2)$ and $(2^{c_1-c_2}q_1+q_2)$ is an odd integer, we have $a_{E+N} = c_2$ and so, $a_E > a_{E+N}$. Thus, $a_E \neq a_{E+N}$.

Case 3 ($c_1 < c_2$). Recall that $E = 2^{c_1}q_1$ is the smallest even integer within an arbitrary segment of the sequence $(a_n)_{n\geq1}$, and so, $N = 2^{c_2}q_2 \geq 4$. Note that one can always find a positive odd integer j such that $2^{c_1}j \leq N - 2$ because, upon simplifying the previous inequality, we find $j \leq 2^{c_2-c_1}q_2 - \frac{1}{2^{c_1-1}}$, and because $c_2 \geq c_1 + 1$ with $q_2 \geq 1$, one has that $2^{c_2-c_1}q_2 - \frac{1}{2^{c_1-1}} \geq 2 - 1 = 1$. Thus, if we let $E_j = E + 2^{c_1}j$, then E_j will be an even integer within the arbitrary left segment of the sequence $(a_n)_{n\geq1}$ having N terms. Furthermore, because $j \leq 2^{c_2-c_1}q_2 - \frac{1}{2^{c_1-1}}$ with $1 \leq c_1 < c_2$, observe that the largest odd integer j is $2^{c_2-c_1}q_2 - 1$.

We now consider the integer pairs $(E_j, E_j + N)$ for $j = 1, 3, \ldots, 2^{c_2-c_1}q_2 - 1$, and the values of a_{E_j} and a_{E_j+N}. Observe that

$$E_j = 2^{c_1}(q_1 + j), \tag{3.1}$$

$$E_j + N = 2^{c_1}(q_1 + j + 2^{c_2-c_1}q_2), \tag{3.2}$$

with $c_2 - c_1 \geq 1$, and so, $q_1 + j$ and $q_1 + j + 2^{c_2-c_1}q_2$ are positive even integers. Consider now a partition of the odd positive integers q_1 into the following residue classes modulo $2^{c_2-c_1}q_2$, that is $q_1 \equiv r \pmod{2^{c_2-c_1}q_2}$ for one and only one $r = 1, 3, 5, \ldots, 2^{c_2-c_1}q_2 - 1$. If $q_1 \equiv r \pmod{2^{c_2-c_1}q_2}$, then $q_1 = r + (i-1)2^{c_2-c_1}q_2$ for some $i \in \mathbb{N}$, and so, for $j = 2^{c_2-c_1}q_2 - r$, observe from (3.1) and (3.2) that

$$E_j = 2^{c_1}((r + (i-1)2^{c_2-c_1}q_2) + (2^{c_2-c_1}q_2 - r)) = i2^{c_2}q_2,$$

$$E_j + N = 2^{c_1}((r + (i-1)2^{c_2-c_1}q_2) + (2^{c_2-c_1}q_2 - r) + 2^{c_2-c_1}q_2) = (i+1)2^{c_2}q_2.$$

Because i and $i + 1$ are of opposite parity, we can conclude $a_{E_j} \neq a_{E_j+N}$. □

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