SELF-SIMILAR STRUCTURE OF $\mathcal{P}$-POSITIONS OF THE GAME EUCLID

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Abstract. Euclid is a Nim-type combinatorial game in which the game moves can be thought of as traversing branches of the Calkin–Wilf tree. In this paper, we show that the $\mathcal{P}$-positions of Euclid exhibit a self-similar structure within the tree, and we relate these $\mathcal{P}$-positions to the Fibonacci numbers. Additionally, we identify the game positions that require the maximum number of moves in optimal play, locate the vertices corresponding to these moves in the Calkin–Wilf tree, and relate them to the integer pairs that require the maximum number of steps in the Euclidean algorithm.

1. Introduction

Cole and Davie [5] introduced the game Euclid, an alternating two-player token-removal game like Nim except that there are exactly two piles of tokens. When faced with positive integer piles of m and n tokens (represented by (m, n) and called a position), a player's turn consists of removing a positive multiple of the smaller number from the larger pile. When initially introduced, a valid move required leaving a nonnegative number of tokens. A subsequent version of the game, proposed by Grossman [6] in the problems section of *Mathematics Magazine* and analyzed here, requires a positive number of tokens to remain. Specifically, if $m > n$, then the player can remove any positive integer multiple k of n from m such that $m - nk > 0$, resulting in the position $(m - nk, n)$. A player loses when there is no valid move for the player to make. Euclid gets its name because subtracting the maximum number of the smaller value from the larger is a repeated step of the Euclidean algorithm, which is used to find the greatest common divisor of the two integers. A consequence of the Euclidean algorithm is that the game ends with the two piles each having $\gcd(m, n)$ tokens.

In [5], Cole and Davie solve Euclid by determining which positions (m, n) are winning for the next player to move (the $\mathcal{N}$-positions, in combinatorial game theory parlance) and which are winning for the previous player who just moved (the $\mathcal{P}$-positions). Cole and Davie's $\mathcal{P}$-positions are the same as those from Grossman's variation of Euclid, which was solved by Straffin [13]. Deciding whether (m, n) is an $\mathcal{N}$- or $\mathcal{P}$-position is achieved by comparing m/n to the golden ratio. By thinking of positions (m, n) in the plane, Lengyel [9] provided a proof of which (m, n) are $\mathcal{N}$- and $\mathcal{P}$-positions and showed that the $\mathcal{P}$-positions form a cone. Hofmann, Schuster, and Steuding [7] showed how Euclid game play may be viewed as moves along branches in the Calkin–Wilf tree [4].

In this paper, we build on these previous results after first reviewing the relationship between $\mathcal{N}$- and $\mathcal{P}$-positions and the golden ratio (Section 2) and recalling how game play of Euclid relates to the Calkin–Wilf tree (Section 3). Then, in Section 4, we show that $\mathcal{P}$-positions of Euclid have self-similar structure within the Calkin–Wilf tree and count the types of $\mathcal{P}$-positions in each row of the tree. The $\mathcal{P}$-positions form arches in the Calkin–Wilf tree and, in Section 5, we determine the number of tokens in every $\mathcal{P}$-position writing them as a linear combination of Fibonacci numbers. Positions (F_n, F_{n+1}) all require a maximum number of possible moves in Euclid. In Section 6, we determine all $\mathcal{N}$- and $\mathcal{P}$-positions that require a maximum number of moves in optimal play and show that these positions are also part of cones in the plane, which in part refines the cone picture of Lengyel [9].

2. Optimal Play

Suppose the game starts with board position $(7, 2)$. Then, the first player to go can leave piles $(5, 2)$, $(3, 2)$ or $(1, 2)$. Is there a best move? And, if so, which is it? Cole and Davie [5] proved that the next player to play from position (m, n) with $m > n$ has a winning strategy if and only if $m/n > (1+\sqrt{5})/2 = \varphi \approx 1.61803$, the golden ratio. In combinatorial game theory, such a position is called an $\mathcal{N}$-position, because the next player to move wins under optimal play. If the next player to move cannot win under optimal play from (m, n), then (m, n) is referred to as a $\mathcal{P}$-position, because the previous player wins under optimal play. The position $(7, 2)$ is an $\mathcal{N}$-position, as evidenced by $7/2 > \varphi$. The optimal play is to take 4 tokens away from the 7, leaving $(3, 2)$. The position $(3, 2)$ is a $\mathcal{P}$-position because $3/2 < \varphi$. Taking away 2 or 6 tokens is not optimal because $(5, 2)$ and $(1, 2)$ are $\mathcal{N}$-positions, as $5/2 > \varphi$ and $2/1 > \varphi$, respectively.

Lengyel [9] used a cone to visualize the $\mathcal{N}$-and $\mathcal{P}$-positions and to prove the Cole and Davie result. The $\mathcal{P}$-positions are in the interior of the cone between the rays $y = \varphi x$ and $y = \bar{\varphi}x = (1/\varphi)x$ and the $\mathcal{N}$-positions are outside the cone; these are the white regions in Figure 1. To understand the geometry behind Lengyel's proof, note that $(7, 2)$ is an $\mathcal{N}$-position outside the cone, but can be reduced to $(3, 2)$, inside the cone; this is the yellow region in Figure 1. Such a move from outside to inside the cone is always possible because the length of the segment between $(a\varphi, a)$ and $(a\bar{\varphi}, a)$ is $a(\varphi - \bar{\varphi}) = a$; this follows because φ is a root of $x^2 - x - 1 = 0$. Similarly, once inside the cone, the next player to move must leave the cone; for example, $(5, 4)$ is a $\mathcal{P}$-position and the only move is to leave the cone to $(1, 4)$. Again, this is because the portion of $y = 4$ in the cone has length 4, as in Figure 1. The same logic, but with vertical movement, holds for positions (m, n) with $n > m$.

There are three possible moves from $(14, 4)$, leading to $(10, 4)$, $(6, 4)$ or $(2, 4)$. These are all just twice the possible pile sizes from starting with $(7, 2)$. Without loss of generality, we can restrict our attention to positions (m, n) with m and n relatively prime.

3. The Calkin–Wilf Tree

In [4], Calkin and Wilf used a simple rule to generate a binary tree that contains all positive rational numbers. Hofmann, Schuster, and Steuding [7] explained how play in Euclid can be represented on the Calkin–Wilf tree. We review how to generate the Calkin–Wilf tree and how play of Euclid can be described on the tree, before proving a self-similar relationship between the $\mathcal{P}$-positions of Euclid in the Calkin–Wilf tree in the next section.

The Calkin–Wilf tree is a binary tree with initial vertex $1 = 1/1$. All subsequent vertices in the tree are labeled with a positive rational number given by the parent-child vertex-generating operation in Figure 2. In general, the parent vertex a/b has two children vertices: $a/(a+b)$ and $(a+b)/b$. We will view left children as males and right children as females. Hence, $a/(a+b)$ is a son and $(a+b)/b$ is a daughter of a/b. Repeated use of the operation yields the Calkin–Wilf tree; the first five levels of the tree appear in Figure 3. Notice that right/female vertices are always greater than one and left/male vertices are always less than one. We extend the familial relationship of parent-child to other vertices in the tree, viewing the Calkin–Wilf tree as a genealogical tree. Referring to the relative position of vertices in Figure 3, vertices $8/5$ and $5/7$ are cousins and are two of the grandchildren of vertex $3/2$. Further, vertex $4/3$

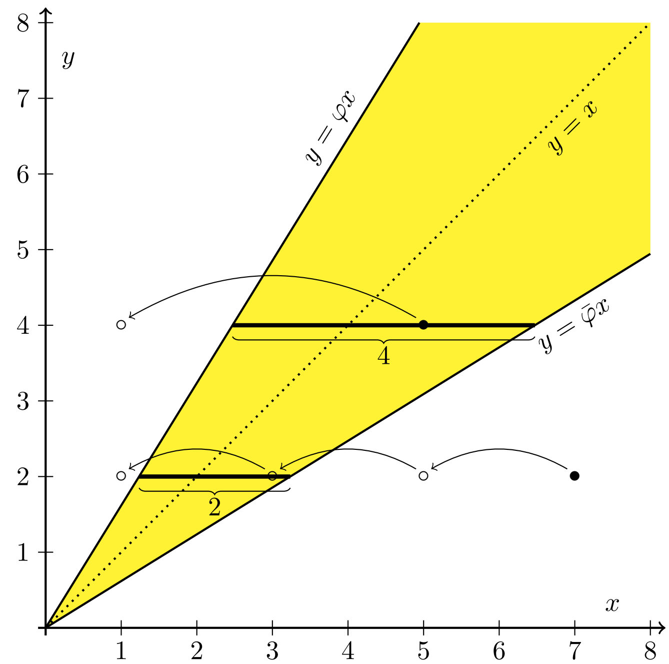

FIGURE 1. The set of $\mathcal{P}$-positions for the game Euclid are the integer pairs in the (yellow-colored) cone between $y = \bar{\varphi}x$ and $y = \varphi x$.

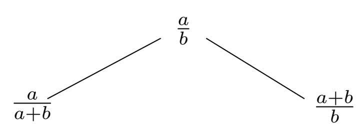

FIGURE 2. The operation to generate the Calkin–Wilf tree.

is the niece of $3/2$ and $4/3$ is the aunt of $5/4$. These relationships will be highlighted later in the paper to describe the self-similar structure of $\mathcal{P}$-positions in the Calkin–Wilf tree.

Importantly, Calkin and Wilf [4] proved that every positive rational number appears exactly once in the tree and appears in reduced form; they also explain how to construct an explicit bijection between the positive integers and positive rational numbers. A position (m, n) in Euclid may be represented by either m/n or n/m in the Calkin–Wilf tree; both are equivalent in the tree, related by symmetry. Because the Calkin–Wilf tree only has rational numbers in reduced form, it further supports the assumption that pile sizes be relatively prime. A move in Euclid can be viewed as traversing up the tree along consecutive left or consecutive right branches. For example, from the position $(7, 2)$ or $7/2$ in the Calkin–Wilf tree, notice that the possible moves $(5, 2)$, $(3, 2)$, and $(1, 2)$ have corresponding fractions $5/2$, $3/2$, and $1/2$ that are all related to one another along right branches in the tree. Because we identify the position (m, n) with rational number m/n, then m/n is referred to as a $\mathcal{P}$-position when (m, n) is a $\mathcal{P}$-position. More details about play on the Calkin–Wilf tree may be found in [7]. Because m and n are relatively prime, play ends in Euclid when $(1, 1)$ is reached. Consequently, $1/1$, the top of the tree, is a $\mathcal{P}$-position.

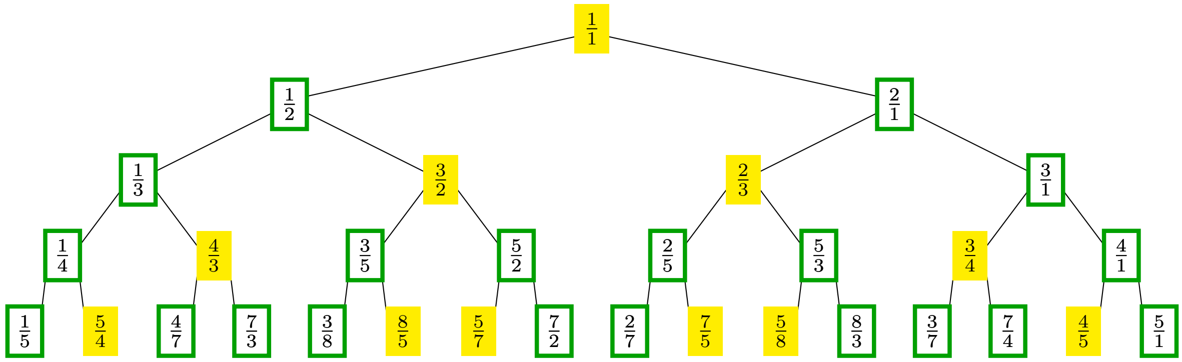

FIGURE 3. The first five levels of the Calkin–Wilf tree. Rational numbers a/b in yellow are $\mathcal{P}$-positions (a, b) in Euclid.

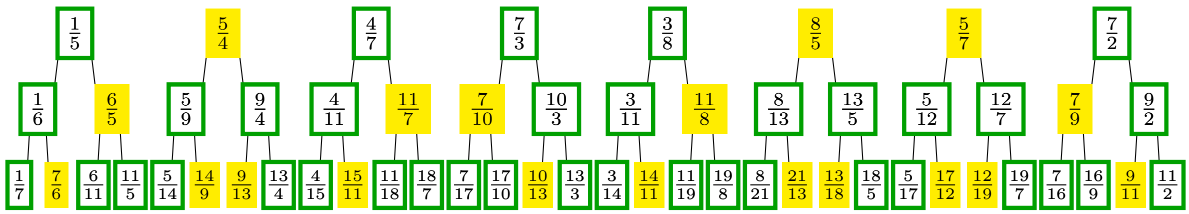

FIGURE 4. The left side of levels 5 through 7 of the Calkin–Wilf tree. Rational numbers a/b in yellow are $\mathcal{P}$-positions (a, b) in Euclid.

4. $\mathcal{P}$-positions in the Calkin–Wilf Tree

In Figure 3, the $\mathcal{P}$-positions appear in yellow (lighter color) and the $\mathcal{N}$-positions are boxed in green (darker color). With only five rows of the tree in Figure 3, patterns of the $\mathcal{P}$-positions may not be obvious. Figure 4 shows the left side of rows 5 through 7 of the Calkin–Wilf tree. Further, the $\mathcal{P}$- and $\mathcal{N}$-positions for the first seven rows of the Calkin–Wilf tree are given in Figure 5. In this figure, the vertices of each row divide the length of row n into 2^{n-1} equal-sized rectangles. A parent's children in this representation are apparent because the children appear in the two rectangles beneath the parent's rectangle.

Patterns are more noticeable in the representation in Figure 5. In particular, the $\mathcal{P}$-positions appear to exhibit self-similarity: every $\mathcal{P}$-position and its descendants have the same $\mathcal{P}/\mathcal{N}$-relationship as the vertex $1/1$ and its descendants (the whole Calkin–Wilf tree). To prove self-similarity, we begin by showing that every $\mathcal{P}$-position (other than $1/1$) is part of an arch (in the yellow/light color) as either one of the top two vertices of an arch (these vertices are cousins to one another) or as part of the branch of an arch. Further, we explain where the tops of arches start in the tree. The relationships between $\mathcal{P}$-positions can be described based on the image of general vertices of the Calkin–Wilf tree in Figure 6. If vertex 1 is a $\mathcal{P}$-position, then vertices 4 and 5 are the top of an arch. The top three rows of an arch appear as vertices 4 – 9. The left branch of an arch consists of right/female vertices and their nieces, grand nieces, great-grand nieces, etc. The right branch of an arch consists of left/male vertices and their nephews, grand nephews, great-grand nephews, etc.

Below we prove how $\mathcal{P}$-positions are related to one another in the Calkin–Wilf tree and how all $\mathcal{P}$-positions except $1/1$ are part of an arch. We start by proving where arches start: for every $\mathcal{P}$-position in row k, there exists the top of an arch in row $k + 2$.

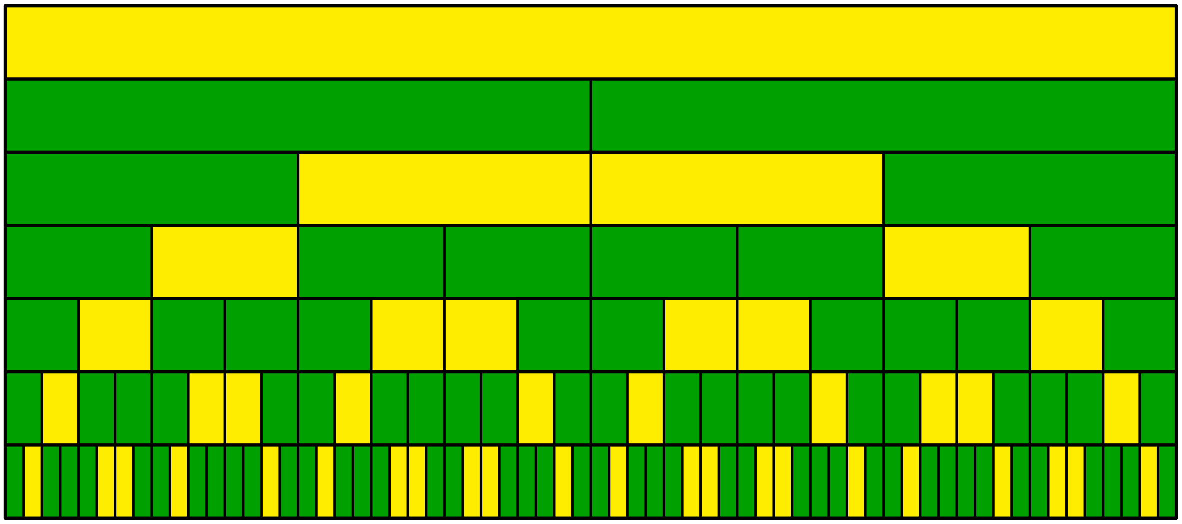

FIGURE 5. A representation of $\mathcal{P}$-positions (yellow/light) and $\mathcal{N}$-positions (green/dark) in the Calkin–Wilf tree.

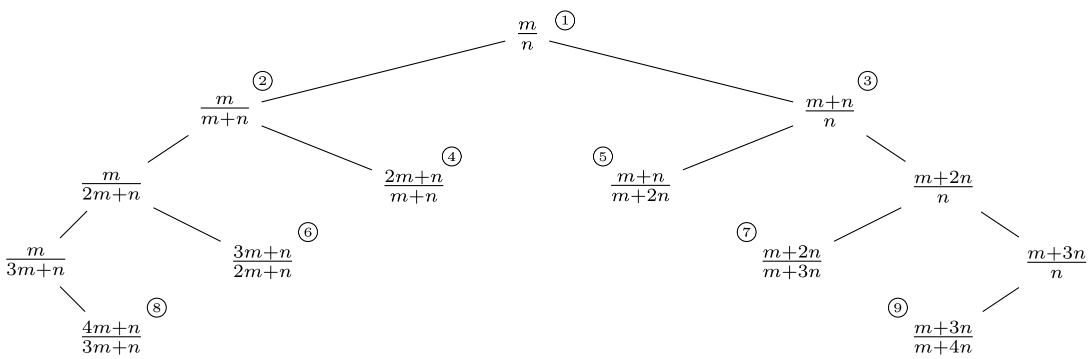

FIGURE 6. Using the positioning of general vertices in the Calkin–Wilf tree to prove the self-similar relationship.

Proposition 4.1. (*Distinguishing the tops of arches.*) *If a vertex V is a $\mathcal{P}$-position, then V's son's daughter and V's daughter's son are $\mathcal{P}$-positions, too.* (*Using the notation in Figure 6, if* (1) *is a $\mathcal{P}$-position, then* (4) *and* (5) *are $\mathcal{P}$-positions, as well.*)

Proof. We will use game play to explain why this proposition holds by showing that if (1) is a $\mathcal{P}$-position, then so is (4). The proof that (5) is a $\mathcal{P}$-position is analogous.

If $\frac{m}{n}$ is a $\mathcal{P}$-position, denoted by (1) in Figure 6, then $\frac{m}{m+n}$ at vertex (2) is an $\mathcal{N}$-position. This follows because a player can take one m away from $m+n$ and leave the opponent with pile sizes of m and n, which is a $\mathcal{P}$-position. It follows that $\frac{2m+n}{m+n}$ is a $\mathcal{P}$-position, denoted by (4) in Figure 6, because the only move, which is to subtract $m+n$ from the pile with $2m+n$ in it, results in an $\mathcal{N}$-position. $\square$

The implication of Proposition 4.1 is that two rows below every $\mathcal{P}$-position is the top of a new arch. The next result proves that every $\mathcal{P}$-position except $1/1$ is part of a branch of an arch with an infinite number of $\mathcal{P}$-positions below it.

Proposition 4.2. (*Establishing branches.*) *If vertex V is a $\mathcal{P}$-position and*

- *V is a right/female vertex, then its niece is also a $\mathcal{P}$-position (in Figure 6, if* 6 *is a $\mathcal{P}$-position, then so is* 8*);*
- *V is a left/male vertex, then its nephew is also a $\mathcal{P}$-position (in Figure 6, if* 7 *is a $\mathcal{P}$-position, then so is* 9*).*

Proof. Because the right/female vertex 6 is a $\mathcal{P}$-position, then $1 < \frac{3m+n}{2m+n} < \varphi$. However, $1 < \frac{4m+n}{3m+n} < \frac{3m+n}{2m+n}$ implies that 8 is a $\mathcal{P}$-position, too. An analogous proof proves that if a left/male vertex 7 is a $\mathcal{P}$-position, then 9 is a $\mathcal{P}$-position, too. □

The following proposition shows that every $\mathcal{P}$-position, other than 1/1, is either part of the top of an arch or part of the branch of an arch.

Proposition 4.3. *If vertex V is a $\mathcal{P}$-position and*

- *V is a right/female vertex, then either its aunt or its male cousin is a $\mathcal{P}$-position, but not both (in Figure 6, if* 4 *is a $\mathcal{P}$-position, then either* 3 *or* 5 *is a $\mathcal{P}$-position, but not both).*
- *V is a left/male vertex, then either its uncle or its female cousin is a $\mathcal{P}$-position, but not both (in Figure 6, if* 5 *is a $\mathcal{P}$-position, then either* 2 *or* 4 *is a $\mathcal{P}$-position, but not both).*

Proof. We will use game play to prove the proposition when V is a right/female vertex. The case for V a left/male vertex is analogous. Interestingly, the implication that one and only one of 3 or 5 is a $\mathcal{P}$-position holds independently of whether or not 4 is a $\mathcal{P}$-position.

From game play, there is only one move from 5 to 3. This follows because there is only one move from piles with sizes $m+n$ and $m+2n$, and that is to remove $m+n$ from $m+2n$, traversing the tree from 5 to 3. Hence, if 3 is a $\mathcal{P}$-position, then the only move from 5 results in a win so that 5 is an $\mathcal{N}$-position. Similarly, if 3 is an $\mathcal{N}$-position, then the only move from 5 has to result in a loss and 5 is a $\mathcal{P}$-position. It follows only one of 3 and 5 is a $\mathcal{P}$-position. □

Because every $\mathcal{P}$-position, besides 1/1, is part of an arch, then the arch structure allows for a way to describe or to navigate to any $\mathcal{P}$-position according to how it is related positionally to the top vertex 1/1. The branch structure from Figure 6 forms a type of nomenclature, just like a sequence of left-right branches describes a vertex in a binary tree from the root vertex. The nomenclature requires some additional notation. For a $\mathcal{P}$-position $\frac{m}{n}$, let $LB^k\left(\frac{m}{n}\right)$ be the vertex that is at the kth level in the left branch of the arch that begins two levels below $\frac{m}{n}$. Define $RB^k\left(\frac{m}{n}\right)$ similarly. Consequently, $LB^1\left(\frac{m}{n}\right)$ (vertex 4) and $RB^1\left(\frac{m}{n}\right)$ (vertex 5) are the tops of the arch below $\frac{m}{n}$ (vertex 1). Extending the pattern from Figure 6, it follows that

$$LB^k\left(\frac{m}{n}\right) = \frac{(k+1)m+n}{km+n} \text{ and } RB^k\left(\frac{m}{n}\right) = \frac{m+kn}{m+(k+1)n}.$$

Any $\mathcal{P}$-position in the tree can be described recursively from the vertex 1/1 in terms of these new functions.

Example 4.4. *As an example of navigating along the arch branches, we describe a path along branches to a particular $\mathcal{P}$-position and then compute its value recursively. The path begins at 1/1 and then goes to the third level of the left branch of the arch below 1/1. From this vertex, we go to the second level vertex of the right branch of the arch below it. Now, we go to the* *arch that stems from this vertex and go to the third vertex on the right branch. A new arch starts two levels below this vertex and we are interested in the vertex at the fifth level of the left branch of this new arch. We find the value of this $\mathcal{P}$-position recursively by*

$$LB^5\left(RB^3\left(RB^2\left(LB^3\left(\frac{1}{1}\right)\right)\right)\right) = LB^5\left(RB^3\left(RB^2\left(\frac{5}{4}\right)\right)\right)$$
$$= LB^5\left(RB^3\left(\frac{13}{17}\right)\right) = LB^5\left(\frac{64}{81}\right) = \frac{465}{401}.$$

Taken together, the three propositions, Propositions 4.1, 4.2, and 4.3, prove the following theorem. An implication is that the descendants below any $\mathcal{P}$-position vertex have the same familial structure to $\mathcal{P}$-position descendants of 1/1. This means that the same nomenclature can be used to navigate from a $\mathcal{P}$-position to one of its descendants.

Theorem 4.5. *The $\mathcal{P}$-positions for the game Euclid are self-similar in the Calkin–Wilf tree.*

The self-similar relationship between $\mathcal{P}$-position vertices in the tree leads to a recursive enumeration of the $\mathcal{P}$-positions in row k and an enumeration of the number of arches that have their top two positions in row k.

Corollary 4.6. *Let L_k be the number of $\mathcal{P}$-positions in row k. Then, $L_0 = 1$, $L_1 = 0$, and $L_{k+2} = L_{k+1} + 2L_k$ for $k \geq 0$. That is, $L_k = 2^k/3 + 2(-1)^k/3$.*

Proof. By Proposition 4.1, each $\mathcal{P}$-position in row k contributes two $\mathcal{P}$-positions to row $k+2$. By Proposition 4.2, each $\mathcal{P}$-position in row $k+1$ contributes one $\mathcal{P}$-position to row $k+2$. Thus, the number of $\mathcal{P}$-positions in row k satisfies the recurrence relation $L_{k+2} = L_{k+1} + 2L_k$ for $k \geq 0$.

The corresponding characteristic equation, $x^2 - x - 2 = 0$, of this linear, homogeneous recurrence relation has solutions $x = 2$ and $x = -1$. So, the general solution has the form $L_k = c_0 2^k + c_1(-1)^k$. Using the initial conditions, $L_0 = c_0 + c_1 = 1$ and $L_1 = 2c_0 - c_1 = 0$, to solve for c_0 and c_1 yields $c_0 = 1/3$ and $c_1 = 2/3$. Thus, $L_k = 2^k/3 + 2(-1)^k/3$ is the explicit formula. □

The sequence generated by $L_{k+2} = L_{k+1} + 2L_k$ with initial conditions $L_0 = 1$ and $L_1 = 0$ appears as A078008 in [12].

Corollary 4.7. *The number of arches whose tops are on row k is $T_k = 2^{k-2}/3 + 2(-1)^k/3$.*

Proof. It is an immediate consequence of Proposition 4.1 that a $\mathcal{P}$-position in row $k-2$ generates the top of a new arch in row k. That is, $T_k = L_{k-2}$. By Corollary 4.6, $L_{k-2} = 2^{k-2}/3 + 2(-1)^k/3$ as desired. □

5. Fibonacci Numbers and the Calkin–Wilf Tree

Every row of the Calkin–Wilf tree contains a position that is a pair of consecutive Fibonacci numbers. Specifically, in the left half of the tree, F_{2n+1}/F_{2n+2} appears in row $2n+1$ as an $\mathcal{N}$-position and F_{2n+1}/F_{2n} appears in row $2n$ as the top left of a $\mathcal{P}$-position arch. These pairs with the $\mathcal{P}$-position structure as described in Section 4 enable us to write every position in the corresponding arch as a linear combination of those Fibonacci numbers. First, we write the branch positions as a linear combination of the top positions.

Lemma 5.1. *If vertices $\frac{a}{b}$ and $\frac{c}{d}$ are the left and right positions, respectively, at the top of an arch, then level k of the left branch of the arch is $\frac{(k+1)a-kb}{ka-(k-1)b}$ and level k of the right branch is $\frac{kd-(k-1)c}{(k+1)d-kc}$.*

Proof. The vertex on level k of the left branch of the arch is the niece of the vertex on level $k-1$. Similarly, the vertex on level k of the right branch of the arch is the nephew of the vertex on level $k-1$. Thus, repeated application of the operation that generates the tree starting at $\frac{a-b}{b}$ (the parent of $\frac{a}{b}$) reveals the kth left-branch vertex to be $\frac{(k+1)a-kb}{ka-(k-1)b}$. Repeated application of the operation starting at $\frac{c}{d-c}$ (the parent of $\frac{c}{d}$) yields $\frac{kd-(k-1)c}{(k+1)d-kc}$ on the right branch. □

The vertex F_{2n+1}/F_{2n} in row $2n$ is a right/female vertex because it is greater than one. Thus, it is the top of the left branch of a new arch. The corresponding top of the right branch is then $F_{2n}/(F_{2n}+F_{2n-2})$. Each of the $\mathcal{P}$-positions in this arch can be written in terms of these Fibonacci numbers as shown in the following proposition.

Proposition 5.2. *The arch of $\mathcal{P}$-positions whose top left and right positions are F_{2n+1}/F_{2n} and $F_{2n}/(F_{2n}+F_{2n-2})$, respectively, is comprised of the positions*

$$\frac{F_{2n+1}+kF_{2n-1}}{F_{2n+1}+(k-1)F_{2n-1}}$$

on level k of the left branch and

$$\frac{F_{2n}+kF_{2n-2}}{F_{2n}+(k+1)F_{2n-2}}$$

on level k of the right branch.

Proof. Lemma 5.1 with $a=F_{2n+1}$, $b=c=F_{2n}$, and $d=(F_{2n}+F_{2n-2})$ implies that level k is comprised of

$$\frac{(k+1)F_{2n+1}-kF_{2n}}{kF_{2n+1}-(k-1)F_{2n}} \quad \text{and} \quad \frac{k(F_{2n}+F_{2n-2})-(k-1)F_{2n}}{(k+1)(F_{2n}+F_{2n-2})-kF_{2n}} \tag{5.1}$$

on the left and right, respectively. By the definition of the Fibonacci numbers, $F_{2n+1}-F_{2n}=F_{2n-1}$. Substituting this into equation (5.1) yields the desired result. □

By symmetry $(F_{2n}+F_{2n-2})/F_{2n}$ and F_{2n}/F_{2n+1} form the top of an arch in the right half of the tree on row $2n$ and the positions in the remainder of the arch can also be written in terms of these Fibonacci numbers. Because the two positions in an arch top are among the grandchildren of a $\mathcal{P}$-position two rows above, Proposition 5.2 can be used to write every arch-top position as a linear combination of Fibonacci numbers. This is explicitly stated in the following corollary.

Corollary 5.3. *The $\mathcal{P}$-position*

$$\frac{F_{2n+1}+kF_{2n-1}}{F_{2n+1}+(k-1)F_{2n-1}}$$

generates the left and right positions at the top of the arch as, respectively,

$$\frac{3F_{2n+1}+(3k-1)F_{2n-1}}{2F_{2n+1}+(2k-1)F_{2n-1}} \text{ and } \frac{2F_{2n+1}+(2k-1)F_{2n-1}}{3F_{2n+1}+(3k-2)F_{2n-1}}.$$

Similarly, the $\mathcal{P}$-position

$$\frac{F_{2n}+kF_{2n-2}}{F_{2n}+(k+1)F_{2n-2}}$$

generates the left and right positions at the top of an arch as, respectively,

$$\frac{3F_{2n}+(3k+1)F_{2n-2}}{2F_{2n}+(2k+1)F_{2n-2}} \text{ and } \frac{2F_{2n}+(2k+1)F_{2n-2}}{3F_{2n}+(3k+2)F_{2n-2}}.$$

Using these tops and the left/right branch formulas in Lemma 5.1, we can describe the branches of all arches in terms of Fibonacci numbers.

6. Maximum Length Game of Optimal Play

The maximum number of moves from any starting position in row k of the Calkin–Wilf tree to the terminal position of $1/1$ in row 0 is k. Thus, optimal play from such a position takes k or fewer moves. Figure 7 shows the number of moves necessary in optimal play from each of the positions in the first five levels of the Calkin–Wilf tree. For example, from a starting position of $4/3$ (the second vertex from the left in row 3 in Figure 3), optimal play takes two moves. Whereas, optimal play from a starting position of $3/5$ (the third vertex from the left in row 3 in Figure 3) takes the maximum number of three moves.

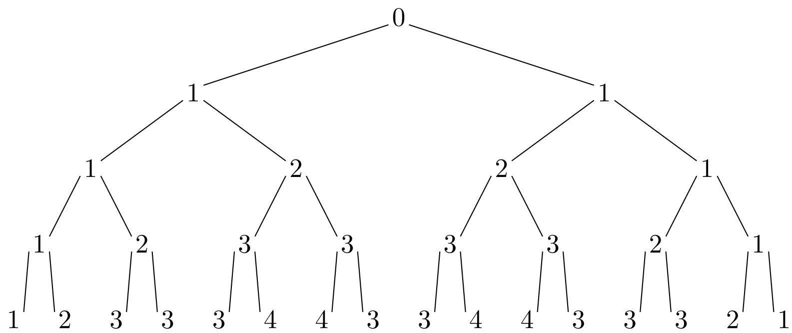

Figure 7. The number of moves in optimal play from each of the positions in the first five levels of the Calkin–Wilf tree.

The number of possible moves from the position (m, n) in Euclid is equal to the quotient in the corresponding step of the Euclidean algorithm. For consecutive Fibonacci numbers, given by $F_0 = 1 = F_1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, necessarily $F_{k+1} = 1 \cdot F_k + r$ where $0 < r < F_k$. It follows that from the position (F_k, F_{k+1}) in Euclid there is exactly one valid move. Thus, a game with initial position (F_k, F_{k+1}), which appears on row k of the tree, requires the maximum number of moves for optimal play. However, these are not the only pairs that require the maximum number of moves.

Proposition 6.1. *For $k \geq 0$, the number of positions in row k of the Calkin–Wilf tree that require k moves in optimal play is $2^{k/2}$ if k is even and $2^{(k+1)/2}$ if k is odd.*

Proof. Moves in maximal optimal play of a combinatorial game alternate between $\mathcal{P}$-positions and $\mathcal{N}$-positions. Because $1/1$ is a $\mathcal{P}$-position, a position in row k in which optimal play requires k moves is a $\mathcal{P}$-position if k is even and an $\mathcal{N}$-position if k is odd.

Because the single position $1/1$ in row 0 is a $\mathcal{P}$-position, both positions, $1/2$ and $2/1$ in row 1 are $\mathcal{N}$-positions. Thus, the claim holds for $k = 0$ and $k = 1$ as $2^{0/2} = 1$ and $2^{(1+1)/2} = 2$.

Suppose k is even and $2^{k/2}$ positions in row k require k moves in optimal play. Let vertex V be such a position, which is necessarily a $\mathcal{P}$-position. Both its children are $\mathcal{N}$-positions and take $k+1$ moves in optimal play. Thus, $2(2^{k/2}) = 2^{(k+2)/2}$ positions in row $k+1$ require $k+1$ moves in optimal play.

Now suppose k is odd and $2^{(k+1)/2}$ positions in row k require k moves in optimal play. Let vertex V be such a position, which is necessarily an $\mathcal{N}$-position. If V is a left/male vertex, then its parent is a $\mathcal{P}$-position and the only valid move from V's daughter is to V. So V's daughter in row $k+1$ requires $k+1$ moves in optimal play. Furthermore, moving from V's son to V's parent is valid, making V's son an $\mathcal{N}$-position in row $k+1$ requiring only k moves in optimal play.

If V is a right/female vertex, then its parent is a $\mathcal{P}$-position and the only valid move from V's son is to V. So V's son in row $k+1$ requires $k+1$ moves in optimal play. Furthermore, moving from V's daughter to V's parent is valid, making V's daughter an $\mathcal{N}$-position in row $k+1$ requiring only k moves in optimal play. Thus, there are $2^{(k+1)/2}$ positions in row $k+1$ that require $k+1$ moves in optimal play. $\square$

The vertices in the Calkin–Wilf tree that require the maximum number of moves in optimal play can be located by following paths in which every odd numbered move is followed by a move in the opposite direction. The following proposition formalizes this idea.

Proposition 6.2. *Every Euclid position that requires the maximum number of moves in optimal play can be found in the Calkin–Wilf tree starting at the* $1/1$ *position and walking along paths such that the* $2n-1$ *and* $2n$ *steps, starting with* $n = 1$*, are either LR or RL. For example, LRLR and LRRL lead to maximum-optimal-play positions, but LRLL does not.*

Proof. The $2n-1$ step on a walk, starting from $1/1$, is a step from an even numbered row to an odd numbered row in the tree. Thus, in a path to a position that requires the maximum number of moves in optimal play, this step from an even numbered row to an odd numbered row is a step from a $\mathcal{P}$-position, say vertex U, to an $\mathcal{N}$-position, say vertex V. Similarly, the $2n$ step on a walk is from an $\mathcal{N}$-position, vertex V, to a $\mathcal{P}$-position, vertex W. If both the steps have the same direction, LL or RR, then a step from vertex U to vertex W is possible. However, this contradicts vertex W being a $\mathcal{P}$-position. Thus, to require the maximum number of moves in optimal play, the $2n-1$ and $2n$ steps must have opposite direction. $\square$

Similar to the description that $\mathcal{N}$-positions (m,n) with $m > n$ satisfy $m/n > (1+\sqrt{5})/2 = \varphi$, we show that a winning strategy requires the maximum number of moves in optimal play if and only if $(\varphi+1) > m/n > \varphi$.

Proposition 6.3. *The next player to play from position* (m,n) *with* $m > n$ *has a winning strategy that requires the maximum number of moves only if* $(\varphi+1) > m/n > \varphi$*.*

Proof. If (m,n) with $(\varphi+1)n < m$ is an $\mathcal{N}$-position requiring the maximum number of moves in optimal play, then $(m-n,n)$ must be a $\mathcal{P}$-position. But, $(\varphi+1)n < m$ implies $\varphi n < m-n$. Thus, $(m-n,n)$ is outside of the $\mathcal{P}$-position cone. $\square$

By symmetry, Proposition 6.3 implies the following corollary.

Corollary 6.4. *The next player to play from position* (m,n) *with* $n > m$ *has a winning strategy that requires the maximum number of moves only if* $1/(\varphi+1) < m/n < 1/\varphi$*.*

Proposition 6.5. *The* $\mathcal{P}$*-positions* (m,n) *that require the maximum number of moves in optimal play satisfy*

$$\frac{\varphi+2}{\varphi+1} < \frac{m}{n} < \varphi \ \text{ or } \ \frac{1}{\varphi} < \frac{m}{n} < \frac{\varphi+1}{\varphi+2}.$$

Proof. Suppose (m,n) is a $\mathcal{P}$-position that requires the maximum number of moves in optimal play and satisfies $(\varphi+1)/(\varphi+2) < m/n < (\varphi+2)/(\varphi+1)$.

If $m < n < [(\varphi+2)/(\varphi+1)]m$, then the only valid move is to $(m, n-m)$. But, $n < [(\varphi+2)/(\varphi+1)]m$ implies $n-m < [1/(\varphi+1)]m$, which means by Proposition 6.3, $(m, n-m)$ is an $\mathcal{N}$-position that does not require the maximum number of moves in optimal play. This is a contradiction.

If $[(\varphi+1)/(\varphi+2)]m < n < m$, then the only valid move is to $(m-n,n)$. But multiplying $[(\varphi+1)/(\varphi+2)]m < n$ by $\varphi+2$ yields $(\varphi+1)m < (\varphi+2)n$. That is, $(\varphi+1)(m-n) < n$. Thus, by Corollary 6.4, $(m-n,n)$ is an $\mathcal{N}$-position that does not require the maximum number of moves in optimal play. This is a contradiction. □

Paralleling Lengyel's approach [10] to visualizing the Sprague–Grundy values for Euclid, the positions that require the maximum number of moves in optimal play correspond to cones in the plane. The $\mathcal{N}$-positions that require the maximum number of moves in optimal play are located in the cone between $y=[1/(\varphi+1)]x$ and $y=(1/\varphi)x$ or the cone between $y=\varphi x$ and $y=(\varphi+1)x$; see Figure 8. The $\mathcal{P}$-positions that require the maximum number of moves in optimal play are located in the cone between $y=(1/\varphi)x$ and $y=[(\varphi+1)/(\varphi+2)]x$ or the cone between $y=[(\varphi+2)/(\varphi+1)]x$ and $y=\varphi x$. For example, $(5,3)$ is an $\mathcal{N}$-position requiring the maximum number of moves in optimal play as $\varphi<5/3<\varphi+1$. Furthermore, the only valid move from $(5,3)$ is to $(2,3)$, which is a $\mathcal{P}$-position satisfying $\bar{\varphi}<2/3<(\varphi+1)/(\varphi+2)$. This too has a single valid move to $(2,1)$, which in turn also leaves one valid move to the final position $(1,1)$. Similarly, $(7,5)$ is a $\mathcal{P}$-position requiring the maximum number of moves in optimal play as $(\varphi+2)/(\varphi+1)<7/5<\varphi$. The only valid move is to $(2,5)$, which is an $\mathcal{N}$-position satisfying $1/(\varphi+1)<2/5<\bar{\varphi}$.

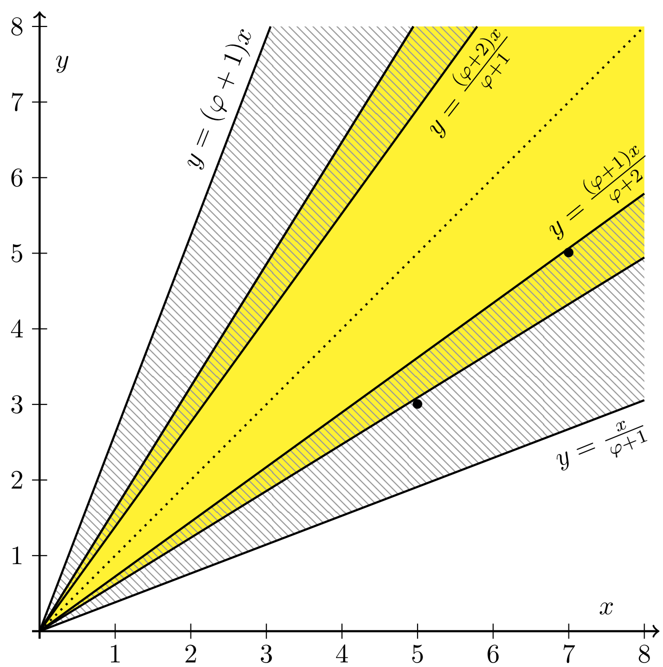

FIGURE 8. As in Figure 1, $\mathcal{P}$-positions are in the yellow cone and $\mathcal{N}$-positions are outside the cone. The integer pairs in the plane between $y=\bar{\varphi}x$ and $y=[(\varphi+1)/(\varphi+2)]x$, as well as the integer pairs between $y=[(\varphi+2)/(\varphi+1)]x$ and $y=\varphi x$ are the set of $\mathcal{P}$-positions for which optimal play of the game Euclid requires the maximum number of moves; these cones are yellow with diagonal lines in gray. The integer pairs in the plane between $y=x/(\varphi+1)$ and $y=\bar{\varphi}x$, as well as the integer pairs between $y=\varphi x$ and $y=(\varphi+1)x$ are the set of $\mathcal{N}$-positions for which optimal play requires the maximum number of moves; these cones have a white background with gray lines.

It is well known that determining the greatest common divisor of consecutive Fibonacci numbers, $\gcd(F_n, F_{n+1})$, requires n steps in the Euclidean algorithm, cf. [2]. Thus, as mentioned previously, the corresponding positions in the game Euclid require n moves in optimal play. However, not every pair of integers that requires the maximum number of steps in the algorithm also requires the maximum number of moves in optimal play.

Hopkins and Tangboonduangjit [8] prove that for $n > 0$, there are n positive integer pairs (u, v) where $F_{n+1} \leq u < F_{n+2}$ and $v \leq u$ such that the Euclidean algorithm requires n steps to determine $\gcd(u, v)$. It should be noted that this result is restated here in terms of the Fibonacci sequence starting with $F_0 = 1$, $F_1 = 1$, whereas in [8] the result was stated in terms of $F_0 = 0$, $F_1 = 1$. The corresponding pairs (u, v) for $n = 1$ to 7 are listed in Table 1. Hopkins and Tangboonduangjit [8] further show that for a given n, the continued fraction expansions of these u/v are $[1^{n-1}, 2]$, $[1, 2, 1^{n-3}, 2]$, $[1, 1, 2, 1^{n-4}, 2]$, $\ldots$, $[1^{n-2}, 2, 2]$, $[1^{n-1}, 3]$, where 1^k represents k consecutive 1s. Notice that the first pair corresponds to F_{n+1}/F_n and the others are derived from adding 1 to one of the coefficients of the continued fraction from the second to the nth entries in the list notation. That is, the continued fraction expansions corresponding to the integer pairs that take the maximum number of steps in the Euclidean algorithm are $[1^{n-1}, 2]$, $[1^i, 2, 1^{n-2-i}, 2]$ for $i = 1$ to $n - 2$, and $[1^{n-1}, 3]$.

n	(u, v)
1	$(2,1)$
2	$(3,2)$, $(4,3)$
3	$(5,3)$, $(7,4)$, $(7,5)$
4	$(8,5)$, $(11,7)$, $(11,8)$, $(12,7)$
5	$(13,8)$, $(18,11)$, $(18,13)$, $(19,11)$, $(19,12)$
6	$(21,13)$, $(29,18)$, $(29,21)$, $(30,19)$, $(31,18)$, $(31,19)$
7	$(34,21)$, $(47,29)$, $(47,34)$, $(49,30)$, $(49,31)$, $(50,29)$, $(50,31)$

TABLE 1. The pairs (u, v) with $F_{n+1} \leq u < F_{n+2}$ and $v \leq u$ such that the Euclidean algorithm requires n steps for $n = 1$ to 7.

Lengyel [9] and Hofmann, Schuster, and Steuding [7] relate positions in Euclid by their continued fraction representations. In the context of the Calkin–Wilf tree, Hofmann, Schuster, and Steuding [7] explain that a reduced fraction with continued fraction expansion $[a_0, a_1, \ldots, a_m]$ can be located in the tree following from left to right the path

$$L^{a_m - 1} R^{a_{m-1}} \cdots L^{a_1} R^{a_0} \quad \text{if } m \text{ is odd, and}$$

$$R^{a_m - 1} L^{a_{m-1}} \cdots L^{a_1} R^{a_0} \quad \text{if } m \text{ is even,}$$

where L^k represents k steps to the left and R^k represents k steps to the right. Thus, the continued fraction expansion $[1^{n-1}, 2]$ corresponds to the path $LR \cdots LR$ if n is even and $RL \cdots RL$ if n is odd. In either case, the path satisfies the conditions in Proposition 6.2 and the corresponding position in Euclid requires the maximum number of moves in optimal play. The continued fraction expansion $[1^{n-1}, 3]$ corresponds to the path $LLRL \cdots R$ if n is even and $RRLR \cdots R$ if n is odd. In either case, the path does not satisfy the conditions in Proposition 6.2 and the corresponding position in Euclid does not require the maximum number of moves in optimal play. If n is even, then $[1^i, 2, 1^{n-2-i}, 2]$ for $i = 1$ to $n-2$ corresponds to the path

$$L\underbrace{RL\cdots RL}_{n-2-i}RR\underbrace{LR\cdots LR}_{i} \quad \text{if } i \text{ is even, and}$$
$$L\underbrace{RL\cdots R}_{n-2-i}LL\underbrace{RL\cdots R}_{i} \quad \text{if } i \text{ is odd.}$$

The latter does not satisfy the conditions of Proposition 6.2, whereas the former does because $n-2-i$ is odd when i is odd and $n-2-i$ is even when i is even. Similarly, if n is odd, then the corresponding path is

$$R\underbrace{LR\cdots L}_{n-2-i}RR\underbrace{LR\cdots LR}_{i} \quad \text{if } i \text{ is even, and}$$
$$R\underbrace{LR\cdots LR}_{n-2-i}LL\underbrace{RL\cdots R}_{i} \quad \text{if } i \text{ is odd.}$$

Here, the latter satisfies the conditions of Proposition 6.2, whereas the former does not because $n-2-i$ is even when i is odd and $n-2-i$ is odd when i is even.

The above discussion is summarized as the following proposition.

Proposition 6.6. *The continued fraction expansions* $[1^{n-1}, 2]$ *and* $[1^i, 2, 1^{n-2-i}, 2]$*, where* n *and* i *have the same parity, correspond to the integer pairs that take the maximum number of steps in the Euclidean algorithm and the maximum number of moves in optimal play.*

The continued fraction expansions $[1^{n-1}, 3]$ *and* $[1^i, 2, 1^{n-2-i}, 2]$*, where* n *and* i *have opposite parity, correspond to integer pairs that take the maximum number of steps in the Euclidean algorithm, but do not require the maximum number of moves in optimal play.*

7. A Further Extension for Future Work

For the game Euclid, we determined the location of all of the $\mathcal{P}$-positions in the Calkin–Wilf tree. A natural follow-up question is to consider the location in the tree of $\mathcal{N}$-positions with different Sprague–Grundy values. Nivasch [11] determined the Sprague–Grundy values for Grossman's variation of Euclid, whereas Cairns, Bao, and Lengyel [3] did the same for the Cole–Davie variation of Euclid, comparing it to the value for the Grossman variation. From their work, the $\mathcal{N}$-positions with a fixed Sprague–Grundy value also are contained in cones; see Nivasch [11, Figure 1] for Grossman's variation and Lengyel [10] for the Cole–Davie version. This suggests that describing the location of $\mathcal{N}$-positions with a particular Sprague–Grundy value in the Calkin–Wilf tree may also be interesting. We leave this for future work.

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