H-606 Proposed by Mario Catalani, University of Torino, Italy

Let us consider, for a nonnegative integer $n$, the following sum

$$S_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{n-k}{2} \right) - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left( \frac{n-1-k}{2} + 1 \right).$$

A summation with a negative upper limit is taken to be equal to zero. Express $S_n$ both in closed form and as a recurrence.

H-607 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let $n$ be a positive integer greater than or equal to 3. Evaluate the sum

$$\sum_{i=1}^{n} \left[ \left( \frac{F_{i+1} - F_{i-1}}{F_{i+2} - F_{i-2}} \right)^{n-2} \prod_{j \neq i} \left( 1 - \frac{F_{j+2} - F_{j-2}}{F_{i+2} - F_{i-2}} \right)^{-1} \right].$$

H-608 Proposed by Mario Catalani, University of Torino, Italy

Let $P_n$ denote the Pell numbers

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1.$$

Find

$$\lim_{n \to \infty} \prod_{k=1}^{n} \left( 1 + \frac{1}{\sqrt{2P_{2k}^2 + 1}} \right).$$
Consider the generalized Fibonacci and Lucas polynomials:

\[ F_{n+1}(x, y) = xF_n(x, y) + yF_{n-1}(x, y), \quad F_0(x, y) = 0, \quad F_1(x, y) = 1; \]
\[ L_{n+1}(x, y) = xL_n(x, y) + yL_{n-1}(x, y), \quad L_0(x, y) = 2, \quad L_1(x, y) = x. \]

Assume \( y \neq 0, 2x^2 - y \neq 0 \). We will write \( F_n \) and \( L_n \) for \( F_n(x, y) \) and \( L_n(x, y) \), respectively.

Show that:

1. \[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k y^{-2k} F_{3k} = \frac{xF_{2n+1} - yF_{2n} + (-x)^{n+2} F_n + (-x)^{n+1} yF_{n-1}}{y^n(2x^2 - y)};
\]
2. \[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k y^{-2k} L_{3k} = \frac{xL_{2n+1} - yL_{2n} + (-x)^{n+2} L_n + (-x)^{n+1} yL_{n-1}}{y^n(2x^2 - y)}.
\]

Solution by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci and Lucas polynomials by

\[ F_0(x) = 0, \quad F_1(x) = 1, \quad \text{and} \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \quad \text{for} \ n \geq 1, \]
\[ L_0(x) = 2, \quad L_1(x) = x, \quad \text{and} \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \quad \text{for} \ n \geq 1, \]

respectively. It is known (see [1]) that

\[ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k F_{3k} = \frac{xF_{2n+1} - yF_{2n} + (-x)^{n+2} F_n + (-x)^{n+1} yF_{n-1}}{2x^2 - 1}; \]

and

\[ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k L_{3k} = \frac{xL_{2n+1} - yL_{2n} + (-x)^{n+2} L_n + (-x)^{n+1} yL_{n-1}}{2x^2 - 1}. \]

Simple induction arguments show that, for all integers \( n, \) \( F_n = F_n(x, y) = (\sqrt{y})^{n-1} F_n(x/\sqrt{y}) \)

and \( L_n = L_n(x, y) = (\sqrt{y})^n L_n(x/\sqrt{y}) \), where \( \sqrt{y} \) can be any of the two possible square roots.
of $y$. Now, it is easily verified that 1 follows from (1) when replacing $x$ by $x/\sqrt{y}$ and dividing the resulting equation by $\sqrt{y}$, and that 2 follows from (2) with $x$ replaced by $x/\sqrt{y}$.


Also solved by Paul Bruckman, Kenneth Davenport, Walther Janous, Vincent Mathe and the proposer.

Binomial Coefficients and Pell Numbers

**H-595** Proposed by José Díaz-Barrero & Juan Egozcue, Barcelona, Spain

(Vol. 41, no. 1, February 2003)

Let $\ell$, $n$ be positive integers. Prove that

$$\sum_{k=0}^{n} \binom{k + \ell + 1}{k + 1} \left\{ \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P_n^{j-k-1} \right\} \leq P^{\ell+1}_n - 1,$$

where $P_n$ is the $n$th Pell number, i.e., $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$.

**Solution by Kenneth Davenport, Frackville, PA**

The inner sum is, by the well-known binomial formula,

$$\sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P_n^{j-k-1} = (-1)^{k+1} \binom{k+1}{k} P_n^k \sum_{j=0}^{k+1} \binom{k+1}{j} (-P_n)^j$$

$$= \frac{(-1)^{k+1}}{P_n^{k+1}} \cdot (1 - P_n)^{k+1} = \left(1 - \frac{1}{P_n}\right)^{k+1}.$$

We are then led to consider the sum

$$\sum_{k=0}^{n} \binom{k + \ell + 1}{k + 1} \left(1 - \frac{1}{P_n}\right)^{k+1}.$$

Substituting $m = k + 1$, we simplify the above expression to get

$$\sum_{m=1}^{n} \binom{m + \ell}{m} \left(1 - \frac{1}{P_n}\right)^m.$$
Next, we make use of the power series

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{n} x^n, \quad \text{for } -1 < x < 1.$$  

We let $x = 1 - 1/P_n$ obtaining that

$$P_n^{\ell+1} - 1 = \sum_{m=1}^{\infty} \binom{m+\ell}{m} \left(1 - \frac{1}{P_n}\right)^m,$$

which implies the desired inequality.

Note that the only feature of $P_n$ that the above proof used is the fact that $P_n > 1$. In particular, the above inequality holds with $P_n$ replaced by any real number $x > 1$.

Also solved by Paul Bruckman, Mario Catalani, Walther Janous, Vincent Mathe, Angel Plaza and Sergio Fáñcon, Ling-Ling Shi, and the proposers.

**Prime Factors of Fibonacci Numbers**

**H-596 Proposed by the Editor**  
*(Vol. 41, no. 2, May 2003)*

A beautiful result of McDaniel (*The Fibonacci Quarterly* 40.1, 2002) says that $F_n$ has a prime divisor $p \equiv 1 \pmod{4}$ for all but finitely many positive integers $n$. Show that the asymptotic density of the set of positive integers $n$ for which $F_n$ has a prime divisor $p \equiv 3 \pmod{4}$ is 1/2. Recall that a subset $\mathcal{N}$ of all the positive integers is said to have an asymptotic density $\lambda$ if the limit

$$\lim_{x \to \infty} \frac{\# \{1 \leq n < x \mid n \in \mathcal{N}\}}{x}$$

exists and equals $\lambda$.

**Solution by the Editor**

Suppose that $n > 3$ is odd. Then $F_n$ is either congruent to 2 modulo 4 or it is odd according to whether $n$ is a multiple of 3 or not. In particular, $F_n$ has odd prime factors and if $q$ denotes anyone of these then reducing the relation

$$L_n^2 - 5F_n^2 = (-1)^n \cdot 4 = -4$$

modulo $q$, we read that $(-4|q) = 1$, therefore $(-1|q) = 1$, and thus $q \equiv 1 \pmod{4}$. Here, for any integer $a$ we used $(a|q)$ to denote the Legendre symbol of $a$ in respect to $q$. This argument shows that $F_n$ is never a multiple of a prime $q \equiv 3 \pmod{4}$ if $n$ is odd, so the set of positive
integers $n$ for which $F_n$ might have a prime divisor $q \equiv 3 \pmod{4}$ is contained in the set of even numbers, and as such it can have asymptotic density at most $1/2$. To prove the result, it suffices to show therefore that most even numbers $n$ have the property that $F_n$ is a multiple of some prime $q \equiv 3 \pmod{4}$. Write $n = 2m$. Assume that there exists a prime factor $p$ of $m$ with $p \equiv 2 \pmod{3}$. Then, $2p \equiv 4 \pmod{6}$. The Fibonacci sequence is periodic modulo 4 with period 6, and if $k \equiv 4 \pmod{6}$, then $F_k \equiv F_4 \pmod{4}$. In particular, $F_{2p} \equiv 3 \pmod{4}$, therefore there must exist a prime factor $q \equiv 3 \pmod{4}$ of $F_{2p}$. Since $2p|n$, it follows that $F_{2p}|F_n$, therefore $q$ divides $F_n$ as well. Thus, if $n = 2m$, then $F_n$ is always divisible by a prime $q \equiv 3 \pmod{4}$, except, eventually, when $m$ is not divisible by any prime number $p \equiv 2 \pmod{3}$. But it is known that these last numbers form a set of asymptotic density zero. In fact, a result of Landau (see [1]) shows that if $x$ is a large positive real number, then the set of positive integers $m \leq x$ such that $m$ is not a multiple of any prime $p \equiv 2 \pmod{3}$ has cardinality $O(x/\sqrt{\log x}) = o(x)$, which completes the proof.


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