ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2004. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-971</u> Proposed by Peter Jeuck, Hewitt, NJ

If A_n is the area of the region under the curve given by

$$y = \sum_{i=0}^{p} a_i x^{p-i}$$

on the interval $[F_n, F_{n+1}]$, show that

$$\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right) = \alpha^{p+1}.$$

B-972 Proposed by Mario Catalani, University of Torino, Torino, Italy

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find an expression for $\mathrm{tr}(A^n)$, where $\mathrm{tr}(\cdot)$ means the trace operator. Let $t_n = \mathrm{tr}(A^n)$. Find a recurrence for t_n.

B-973 Proposed by José Luis Díaz-Barrero & Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ 1 + \sqrt{\frac{F_{2k}}{F_{k+1}^2}} \right\}^{-1} > 2^{n-1}.$$

B-974 Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya Barcelona, Spain

Let n be a positive integer. Prove that

$$\sqrt{\sum_{k=1}^{n} F_{k+1}^2} \le \frac{1}{2n} \left\{ \sqrt{\sum_{k=1}^{n} [F_k + (n^2-1)L_k]^2} + \sqrt{\sum_{k=1}^{n} [(n^2-1)F_k + L_k]^2} \right\}.$$

B-975 Proposed by N. Gauthier, Royal Military College of Canada

For l, m and n nonnegative integers, find closed-form expressions for the following sums:

$$A. \quad \sum_{k=0}^{n} k \binom{n}{k} L_{km+l} L_{mn-km};$$

$$B. \quad \sum_{k=0}^{n} k \binom{n}{k} L_{km+l} F_{mn-km};$$

$$C. \quad \sum_{k=0}^{n} k \binom{n}{k} F_{km+l} L_{mn-km};$$

$$D. \quad \sum_{k=0}^{n} k \binom{n}{k} F_{km+l} F_{mn-km}.$$

SOLUTIONS

Estimate Even Lucas

B-956 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazo, MI (Vol. 41, no. 2, May 2003)

Prove that

$$\frac{1+\sqrt{5}}{4} \leq \sum_{n=0}^{\infty} \frac{1}{L_{2n}} \leq \frac{3}{2}.$$

Solution by Paul S. Bruckman, Sointula, BC, Canada.

The following paper [PB] is cited:

P. Bruckman. "On the Evaluation of Certain Infinite Series by Elliptic Functions." *The Fibonacci Quarterly* **15.4** (1977): 293-310.

In [PB], the author showed, inter alia, that $\sum_{n=1}^{\infty} 1/L_{2n} = 0.56618$, approximately. Therefore, $S \equiv \sum_{n=0}^{\infty} 1/L_{2n} = 1.06618$, approximately. The approximate value of $(1+5^{1/2})/4 = \alpha/2$ is 0.80902. Thus, it is easily seen that the desired inequalities hold (strictly; moreover, they could be considerably strengthened at either end).

Paul Bruckman gave two solutions. L.A.G. Dresel obtained the same estimate using a different method.

Also solved by Gordial Arora and Donna Stutson (jointly), Brian D. Beasley, Mario Catalani, Charles Cook, L.A.G. Dresel, Steve Edwards and James Whitenton (jointly), Pentti Haukkanen, Russell Hendel, Ernst Herrmann, Walther Janous, Harris Kwong, J.F. Morrison, H.-J. Seiffert, Jaroslav Seibert, J. Suck, Bing Wang, and the proposer.

Two Lucas Equalities

B-957 Proposed by Muneer Jebreel, Jerusalem, Israel (Vol. 41, no. 2, May 2003)

For $n \geq 1$, prove that

(a) $L^2_{2^n+3} + 4 = 4L_{2^{n+1}+3} + L^2_{2^n}$

and

(b) $L^2_{2^n+6} = 4 + 4L_{2^{n+1}+9} + L^2_{2^n+3}$.

Solution by L.A.G. Dresel, Reading, England.

For $n \geq 1$, (a) and (b) are special cases of the more general result

$$(L_{t+3})^2 + 4(-1)^t = 4L_{2t+3} + (L_t)^2,$$

which we shall now prove.

We have $(L_t)^2 = (\alpha^t + \beta^t)^2 = L_{2t} + 2(-1)^t$, so that $(L_{t+3})^2 = L_{2(t+3)} - 2(-1)^t$, giving $(L_{t+3})^2 - (L_t)^2 + 4(-1)^t = L_{2t+6} - L_{2t}$.

It only remains to show that $L_{2t+6} - L_{2t} = 4L_{2t+3}$.

Now we have

$$L_{k+3} - 2L_k = L_{k+2} + L_{k+1} - 2L_k = L_{k+1} + L_{k-1},$$

and

$$L_{k-3} + 2L_k = L_{k-1} - L_{k-2} + 2L_k = L_{k+1} + L_{k-1}$$

so that by subtracting we obtain the recurrence $L_{k+3} - 4L_k - L_{k-3} = 0$.

Putting $k = 2t + 3$ completes the proof.

Also solved by Gurdial Arora and Donna Stutson (jointly), Paul S. Bruckman, Mario Catalani, Charles Cook, Kenneth B. Davenport, Ovidiu Furdui, Pentti Haukkanen, Russell Hendell, Ernst Herrmann, Walther Janous, Harris Kwong, J.F. Morrison, Maitland A. Rose, H.-J. Seiffert, Jaroslav Seibert, Bing Wang, and the proposer.

The Greatest Common Divisor is $\cdots$

B-958 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain (Vol. 41, no. 2, May 2003)

Find the greatest common divisor of

$$2 + \sum_{k=1}^{n} L_k^2 \text{ and } 3 + \sum_{k=1}^{n} L_k.$$

Solution by Maitland A. Rose, University of South Carolina, Sumter, SC.

Use is made of the well known results:

(i) $\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2, \sum_{k=1}^{n} L_k = L_{n+2} - 3,$ and $(L_n, L_{n+1}) = 1$

(See, for example, pages 77-78 and 203 of *Fibonacci and Lucas Numbers with Applications* by Thomas Koshy)

(ii) If a and b are integers such that $(a, b) = 1$, then $(ab, a + b) = 1$. (See for example pages 187 and 314 of *Equations and Inequalities* by Jiři Herman, Radan Kučera and Jaromlr Šimša)

By results (i), the problem becomes finding $(L_n L_{n+1}, L_{n+2})$. We have $(L_n L_{n+1}, L_{n+2}) = (L_n L_{n+1}, L_n + L_{n+1}) = 1$ by (i) and (ii).

Also solved by Brian Beasley, Paul S. Bruckman, Mario Catalani, Charles Cook, L.A.G. Dresel, Ovidui Furdui, Russell Hendell, Ernst Herrmann, Walther Janous, Harris Kwong, Zhanlan Li, Pentti Haukkanen, Maitland Rose, H.-J. Seiffert, Jaroslav Seibert, and the proposer.

Sum the Sum

B-959 Proposed by John H. Jaroma, Austin College, Sherman, TX (Vol. 41, no. 2, May 2003)

Find the sum of the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{3}{8}+\frac{3}{16}+\frac{8}{32}+\frac{7}{64}+\frac{21}{128}+\frac{15}{256}+\frac{55}{512}+\frac{31}{1024}+\frac{144}{2048}+\frac{63}{4096}+\frac{377}{8192}+\frac{127}{16384}+\cdots$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

We represent the given series by $\sum_{n=1}^{\infty} a_n$. Then

$$\sum_{n=1}^{\infty} a_{2n-1} = 1+\sum_{k=1}^{\infty}\left(\frac{2^k-1}{4^k}\right) = 1+\left(1-\frac{1}{3}\right)=\frac{5}{3}.$$

Also,

$$\sum_{n=1}^{\infty} a_{2n} = \sum_{k=1}^{\infty}\frac{F_{2k}}{2\cdot 4^{k-1}} = \sum_{k=1}^{\infty}\left(\frac{\alpha^{2k}-\beta^{2k}}{2\sqrt{5}\cdot 4^{k-1}}\right) = \frac{1}{2\sqrt{5}}\left(\frac{4\alpha^2}{4-\alpha^2}-\frac{4\beta^2}{4-\beta^2}\right)=\frac{8}{5}.$$

Thus $\sum_{n=1}^{\infty} a_n = 5/3+8/5 = 49/15$.

Also solved by Gurdial Aurora and Donna Stutson (jointly), Paul Bruckman, Mario Catalani, Charles Cook, L.A.G. Dresel, Walther Janous, Kenneth Davenport, Russell Hendell, Harris Kwong, Zhanlan Li, John Morrison, H.-J. Seiffert, James Sellers, Jaroslav Seibert, and the proposer. Two incorrect answers were received.

A Fibonacci Identity

B-960 Proposed by Bob Johnson, Durham University, Durham, England (Vol. 41, no. 2, May 2003)

If $a+b=c+d$, prove that

$$F_aF_b - F_cF_d = (-1)^r[F_{a-r}F_{b-r}-F_{c-r}F_{d-r}]$$

for all integers a, b, c, d and r.

Solution by H.-J. Seiffert, Berlin, Germany

It is known [see A.F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985): 7-20, eqn. (3.32)] that, for all integers n, h, and k,

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^n F_hF_k.$$

Taking $n = c, h = a - c$, and $k = b - c$ gives

$$F_aF_b - F_cF_{a+b-c} = (-1)^c F_{a-c}F_{b-c},$$

and with $n = c - r, h = a - c$, and $k = b - c$, we obtain

$$F_{a-r}F_{b-r} - F_{c-r}F_{a+b-c-r} = (-1)^{c-r} F_{a-c}F_{b-c}.$$

Combining these identities and using the given condition $a + b - c = d$ give the proposed equation.

Also solved by Gurdial Arora and Sindhu Urrithan (jointly), Paul S. Bruckman, Mario Catalini, J.L. Diás-Barrero and Ó. Ciaurri-Ramirez (jointly), L.A.G. Dresel, Ovidiu Furdui, Ernst Herrmann, Walther Janous, Harris Kwong, Zhanlan Li, John F. Morrison, Maitland Rose, Jaroslav Seibert, Ling-Ling Shi, Bing Wang, and the proposer.

We would like to belatedly acknowledge the solutions to problems B-941 and B-942 by Jaroslav Seibert and Carl Libis; respectively. We apologize for the oversight.