EXTENDING THE BERNOULLI-EULER METHOD FOR FINDING ZEROS OF HOLOMORPHIC FUNCTIONS

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1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}(a_{r-1} \neq 0)$ and $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}(r \geq 1)$ be two sequences of real or complex numbers. The sequence $\{V_n^{(r)}\}_{n \geq -r+1}$ defined by $V_n^{(r)} = \alpha_{-n}$ for $-r+1 \leq n \leq 0$ and the linear recurrence of order r

$$V_{n+1}^{(r)} = a_0 V_n^{(r)} + a_1 V_{n-1}^{(r)} + \cdots + a_{r-1} V_{n-r+1}^{(r)} \quad (n \geq 0) \tag{1.1}$$

is called a *weighted r-generalized Fibonacci sequence*. Such sequences have been extensively studied in the literature (see [6, 10, 11, 13] for example). In this paper we shall refer to such an object as a *sequence of type* (1.1). Such sequences have interested many authors because of their various applications. For example, in numerical analysis some discretization by finite divisions gives such a linear recurrence relation (for example, see [2, 4, 8, 9]).

Sequences of type (1.1) have been generalized in [14, 15] as follows. Let $\{a_j\}_{j \geq 0}$ and $\{\alpha_j\}_{j \geq 0}$ be two sequences of real or complex numbers. The sequence $\{V_j\}_{j \in \boldsymbol{Z}}$ defined by $V_n = \alpha_{-n} (n \leq 0)$ and the linear recurrence of order ∞

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_m V_{n-m} + \ldots \quad (n \geq 0) \tag{1.2}$$

is called an *∞-generalized Fibonacci sequence*. Such sequences have been studied under some hypotheses on the two sequences $\{a_j\}_{j \geq 0}$ and $\{\alpha_j\}_{j \geq 0}$ which guarantee the existence of the terms V_n for every $n \geq 1$ (see [3, 14, 15, 17]). The origin of r- or ∞-generalized Fibonacci sequences goes back to Euler. In [7, Chapter XVII] he discussed Daniel Bernoulli's method of using linear recurrences to approximate zeros of (mainly polynomial) functions.

In this paper, we first study the relationship between a given *polynomial* function and the associated sequence of type (1.1), and then we use it to approximate and find a zero of the polynomial through Bernoulli's method (§2). Our results will be a bit weaker than the usual ones; nevertheless, we have included them in the aim to generalize them to the case of general *holomorphic* functions. In §§3 and 4, this will be carried out through the use of ∞-generalized Fibonacci sequences. These results are very important, since, as far as the authors know, there has been practically no method for approximating or finding a zero of an arbitrary holomorphic function using the coefficients in their power series expansions. Furthermore, in §4, we will discuss the approximation process by using r-generalized Fibonacci sequences with r finite (see [3]), which will enable us to obtain more precise results.

2. BERNOULLI'S METHOD FOR POLYNOMIAL FUNCTIONS

In order to approximate a root of a polynomial $P_r(X)$ of degree r, Bernoulli considered a sequence $\{V_n^{(r)}\}_{n\geq -r+1}$ of type (1.1) such that $P_r(X)$ is its characteristic polynomial. More precisely, he used the initial values $V_0^{(r)} = 1$ and $V_{-1}^{(r)} = \cdots = V_{-r+1}^{(r)} = 0$. It is well known that under certain conditions, if

$$q = \lim_{n\to\infty} \frac{V_{n+1}^{(r)}}{V_n^{(r)}}$$

exists, then it is a root of $P_r(X)$ such that $|q'| \leq |q|$ for any other root q' of $P_r(X)$ (see [8, 9] or [6, Theorem 7], for example). The aim of this section is to establish similar results by using the theory of holomorphic functions.

Let $Q_r(z) = 1 - a_0 z - \cdots - a_{r-2} z^{r-1} - a_{r-1} z^r$ be a complex polynomial of degree $r (r \geq 1, a_{r-1} \neq 0)$, and consider the complex function $f_r(z) = 1/Q_r(z)$. Since $Q_r(0) = 1 \neq 0$, the Taylor expansion of $f_r(z)$ in a disk centred at 0 can be written as

$$f_r(z) = \sum_{n=0}^{\infty} V_n^{(r)} z^n \tag{2.1}$$

for some complex numbers $V_0^{(r)}, V_1^{(r)}, \ldots$. The identity $Q_r(z)f(z) = 1$ implies that

$$V_{n+1}^{(r)} = \sum_{j=0}^{r-1} a_j V_{n-j}^{(r)}$$

for all $n \geq 0$, where $V_0^{(r)} = 1$ and $V_{-1}^{(r)} = \cdots = V_{-r+1}^{(r)} = 0$. Hence, $\{V_n^{(r)}\}_{n\geq -r+1}$ is a sequence of type (1.1) and its characteristic polynomial coincides with $P_r(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-2} X - a_{r-1}$.

Remark 2.1: Conversely, suppose that $\{V_n^{(r)}\}_{n\geq -r+1}$ is a sequence of type (1.1) such that $V_0^{(r)} = 1$ and $V_{-1}^{(r)} = \cdots = V_{-r+1}^{(r)} = 0$. Then we have

$$f_r(z) = \sum_{n=0}^{\infty} V_n^{(r)} z^n = \frac{1}{Q_r(z)},$$

where $Q_r(z) = 1 - a_0 z - \cdots - a_{r-2} z^{r-1} - a_{r-1} z^r$.

The polynomial function Q_r has a root and $Q_r(0) \neq 0$. Hence, the function $f_r = 1/Q_r$ has a Taylor expansion near 0 and it is defined in the open disk of radius

$$R = \min\{|\lambda|;\ \lambda \text{ is a root of } Q_r\}.$$

Note that we always have $0 < R < \infty$. Thus, by using the standard theory of power series (for example, see [1]), we can prove the following (for more details, see the proof of Theorem 3.2 in the next section).

Proposition 2.2: *Let $Q_r(z) = 1 - a_0 z - \cdots - a_{r-2}z^{r-1} - a_{r-1}z^r (a_{r-1} \neq 0)$ be a complex polynomial of degree r. Consider the sequence $\{V_n^{(r)}\}_{n \geq -r+1}$ of type (1.1) whose coefficients and initial values are given by $a_0, a_1, \ldots, a_{r-1}$ and $V_0^{(r)} = 1, V_{-1}^{(r)} = \cdots = V_{-r+1}^{(r)} = 0$ respectively. We suppose that $V_n^{(r)} \neq 0$ for all sufficiently large n. Then the radius of convergence R of the series (2.1) satisifies*

$$\liminf_{n\to\infty} \left| \frac{V_n^{(r)}}{V_{n+1}^{(r)}} \right| \leq R \leq \limsup_{n\to\infty} \left| \frac{V_n^{(r)}}{V_{n+1}^{(r)}} \right|$$

and $R = \min\{|\lambda|_i \lambda$ is a root of $Q_r\}$. In particular, we have $Q_r(Re^{i\theta}) = 0$ for some $\theta \in [0, 2\pi)$, and $R \leq |\mu|$ for all other roots μ of Q_r.

As an immediate corollary, we have the following.

Corollary 2.3: *In the above proposition, if*

$$\Lambda^{(r)} = \lim_{n\to\infty} \left| \frac{V_n^{(r)}}{V_{n+1}^{(r)}} \right|$$

exists, then $\Lambda^{(r)}$ is the smallest among the moduli of the roots of Q_r.

Remark 2.4: As we noted before, if

$$\lambda^{(r)} = \lim_{n\to\infty} \frac{V_n^{(r)}}{V_{n+1}^{(r)}}$$

exists, then actually $\lambda^{(r)}$ itself is a root of Q_r with the smallest modulus (for example, see [6]). In fact, we can easily show that $Q_r(\lambda^{(r)}) = 0$ as follows:

$$\begin{aligned} Q_r(\lambda^{(r)}) &= \lim_{n\to\infty} \left(1 - a_0 \frac{V_n^{(r)}}{V_{n+1}^{(r)}} - a_1 \left(\frac{V_n^{(r)}}{V_{n+1}^{(r)}} \right)^2 - \cdots - a_{r-1} \left(\frac{V_n^{(r)}}{V_{n+1}^{(r)}} \right)^r \right) \\ &= \lim_{n\to\infty} \left(1 - a_0 \frac{V_n^{(r)}}{V_{n+1}^{(r)}} - a_1 \frac{V_n^{(r)}}{V_{n+1}^{(r)}} \frac{V_{n-1}^{(r)}}{V_n^{(r)}} - \cdots - a_{r-1} \frac{V_n^{(r)}}{V_{n+1}^{(r)}} \cdots \frac{V_{n-(r-1)}^{(r)}}{V_{n+1-(r-1)}^{(r)}} \right) \\ &= \lim_{n\to\infty} \left(1 - a_0 \frac{V_n^{(r)}}{V_{n+1}^{(r)}} - a_1 \frac{V_{n-1}^{(r)}}{V_{n+1}^{(r)}} - \cdots - a_{r-1} \frac{V_{n-(r-1)}^{(r)}}{V_{n+1}^{(r)}} \right) \\ &= \lim_{n\to\infty} \frac{V_{n+1}^{(r)} - a_0 V_n^{(r)} - a_1 V_{n-1}^{(r)} - \cdots - a_{r-1} V_{n-(r-1)}^{(r)}}{V_{n+1}^{(r)}} = 0. \end{aligned}$$

Example 2.5: Consider the usual Fibonacci sequence $\{F_{n+1}\}_{n\geq -1}$, which is a sequence of type (1.1) with $r = 2$. In this case, the corresponding polynomial is $Q_2(z) = 1 - z - z^2$. Furthermore, it is well known that

$$\lambda^{(2)} = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{\sqrt{5}-1}{2}.$$

It is easy to verify that λ is the root of Q_2 with the smallest modulus.

Remark 2.6: In the above results, the condition that $V_n^{(r)} \neq 0$ for all sufficiently large n is essential. For example, if r is even and $Q_r(z)$ is a polynomial of z^2, then in the power series expansion of $f_r(z)$, the coefficients $V_n^{(r)}$ with n odd are all zero. Thus we cannot consider $V_n^{(r)}/V_{n+1}^{(r)}$ for even n.

We have a combinatorial expression for sequences of type (1.1) as follows.

Proposition 2.7: *Let $\{V_n^{(r)}\}_{n\geq -r+1}$ be a sequence of type (1.1) whose coefficients and initial values are $a_0, a_1, \dots, a_{r-1}$ and $V_0^{(r)} = 1, V_{-1}^{(r)} = \cdots = V_{-r+1}^{(r)} = 0$ respectively. Then we have*

$$V_n^{(r)} = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\dots k_{r-1}!} a_0^{k_0} a_1^{k_1} \dots a_{r-1}^{k_{r-1}} \tag{2.2}$$

for all $n \geq -r+1$, where $k_0, k_1, \dots, k_{r-1}$ run over nonnegative integers.

Proof: Let us prove the assertion by induction on n. It is easy to see that it is true for $n \leq 0$. Suppose that $n \geq 0$ and that the assertion is true for all integers less than or equal to n. It is easy to see that

$$\sum_{j=0}^{r-1} \frac{(k_0+k_1+\cdots+k_{r-1}-1)!}{k_0!k_1!\dots k_{j-1}!(k_j-1)!k_{j+1}!\dots k_{r-1}!} = \frac{(k_0+\cdots+k_{r-1})!}{k_0!\dots k_{r-1}!}$$

holds, where we ignore the terms corresponding to those j with $k_j = 0$. Then, using this, we see that

$$
\begin{aligned}
V_{n+1}^{(r)} &= \sum_{j=0}^{r-1} a_j V_{n-j}^{(r)} \\
&= \sum_{j=0}^{r-1} a_j \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-j} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\dots k_{r-1}!} a_0^{k_0} a_1^{k_1} \dots a_{r-1}^{k_{r-1}} \\
&= \sum_{j=0}^{r-1} a_j \sum_{k_0+2k_1+\cdots+rk_{r-1}=n+1, k_j \geq 1} \frac{(k_0+\cdots+k_{r-1}-1)!}{k_0!\dots(k_j-1)!\dots k_{r-1}!} a_0^{k_0} \dots a_j^{k_j-1} \dots a_{r-1}^{k_{r-1}} \\
&= \sum_{k_0+2k_1+\cdots+rk_{r-1}=n+1} \sum_{j=0}^{r-1} \frac{(k_0+\cdots+k_{r-1}-1)!}{k_0!\dots(k_j-1)!\dots k_{r-1}!} a_0^{k_0} \dots a_j^{k_j} \dots a_{r-1}^{k_{r-1}} \\
&= \sum_{k_0+2k_1+\cdots+rk_{r-1}=n+1} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\dots k_{r-1}!} a_0^{k_0} a_1^{k_1} \dots a_{r-1}^{k_{r-1}}.
\end{aligned}
$$

This completes the proof. □

Compare the above proposition with [5, 12, 16].

Let us denote the right hand side of the equation (2.2) by $\rho(n,r)$. Then by Corollary 2.3, if

$$\Lambda^{(r)} = \lim_{n\to\infty} \left| \frac{\rho(n,r)}{\rho(n+1,r)} \right|$$

exists, then $(\Lambda^{(r)})^{-1}$ is the largest among the moduli of the roots of the characteristic polynomial $P_r(X)$, and the radius of convergence R of the Taylor series (2.1) of $f_r(z) = 1/Q_r(z)$ coincides with $\Lambda^{(r)}$. Furthermore, if

$$\lambda^{(r)} = \lim_{n\to\infty} \frac{\rho(n,r)}{\rho(n+1,r)}$$

exists, then $\lambda^{(r)}$ is a root of Q_r as we have seen in Remark 2.4. In other words, we can approximate a root of Q_r with the smallest modulus by using $a_0, a_1, \dots, a_{r-1}$ together with the combinatorial formula (2.2).

Remark 2.8: The Taylor expansion of the complex function $f_r(z) = 1/Q_r(z)$ in the open disk $D(0;R)$, with R being as above, is given by

$$f_r(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f_r^{(n)}(0) z^n.$$

Thus, from the expression (2.1) we derive that $f_r^{(n)}(0) = n! V_n^{(r)}$ for all $n \geq 0$.

3. THE BERNOULLI-EULER METHOD FOR HOLOMORPHIC FUNCTIONS

In this section, we show that Bernoulli's method for approximating and finding a root of a polynomial function presented in §2 can be extended to the case of holomorphic functions.

Let $Q(z)$ be a complex function which is holomorphic in a neighbourhood of 0. Let $R_1 > 0$ be the largest positive number such that Q is holomorphic in the open disk $D(0; R_1)$. In order to study the zeros of Q in $D(0; R_1) - \{0\}$, we may only consider the case where Q takes the form

$$Q(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}. \tag{3.1}$$

Since $Q(0) = 1 \neq 0$, $f(z) = 1/Q(z)$ has a Taylor expansion in a certain disk centred at 0, which is of the form

$$f(z) = \sum_{n=0}^{\infty} V_n z^n. \tag{3.2}$$

The identity $Q(z)f(z) = 1$ implies that we have

$$V_{n+1} = \sum_{j=0}^{\infty} a_j V_{n-j}$$

for all $n \geq 0$, where $V_0 = 1$ and $V_{-j} = 0$ and for all $j \geq 1$. Hence, $\{V_n\}_{n \in Z}$ is an ∞-generalized Fibonacci sequence as in (1.2) whose initial values are given by $V_0 = 1$ and $V_{-j} = 0$ for all $j \geq 1$.

Remark 3.1: Conversely, suppose that $\{V_n\}_{n \in \boldsymbol{Z}}$ is a sequence as in (1.2) such that $V_0 = 1$ and $V_{-j} = 0$ for all $j \geq 1$. Then, we have

$$f(z) = \sum_{n=0}^{\infty} V_n z^n = \frac{1}{Q(z)}$$

formally, where $Q(z)$ is given by (3.1).

As a direct generalization of Proposition 2.2, we have the following.

Theorem 3.2: *Let*

$$Q(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}$$

be a holomorphic function defined in a neighbourhood of the origin with radius of convergence $R_1 > 0$. *Consider the sequence* $\{V_n\}_{n \in \boldsymbol{Z}}$ *as in* (1.2) *whose coefficients and initial values are given by* $\{a_j\}_{j \geq 0}$ *and* $V_0 = 1, V_{-j} = 0$ *for all* $j \geq 1$, *respectively. We suppose that* $V_n \neq 0$ *for all sufficiently large* n *and that the radius of convergence* R *of the series* (3.2) *satisfies* $R < R_1$. *Then, we have*

$$\liminf_{n \to \infty} \left| \frac{V_n}{V_{n+1}} \right| \leq R \leq \limsup_{n \to \infty} \left| \frac{V_n}{V_{n+1}} \right|$$

and $R = \min\{|\lambda|; \lambda$ *is a zero of* $Q\}$. *In particular, we have* $Q(Re^{i\theta}) = 0$ *for some* $\theta \in [0, 2\pi)$, *and* $R \leq |\mu|$ *for all other zeros* μ *of* Q.

Proof: It is well known that $R = \left(\limsup_{n\to\infty} \sqrt[n]{|V_n|}\right)^{-1}$ (for example, see [1]). Let L be an arbitrary real number such that $0 < L < \liminf_{n\to\infty} |V_n/V_{n+1}|$. Then there exists an N such that $|V_n/V_{n+1}| > L$ for all $n \geq N$. Therefore, $|V_{N+k}| < |V_N|L^{-k}$ for $k = 1, 2, 3, \ldots$, and hence $\sqrt[N+k]{|V_{N+k}|} < \sqrt[N+k]{|V_N|L^{-k}} = L^{-1}\sqrt[N+k]{|V_N|L^N}$. This implies that $R^{-1} = \limsup_{k\to\infty} \sqrt[N+k]{|V_{N+k}|} \leq L^{-1}$. Since L is arbitrary, we conclude that $\liminf_{n\to\infty} |V_n/V_{n+1}| \leq R$. By a similar argument, we can show that $R \leq \limsup_{n\to\infty} |V_n/V_{n+1}|$.

For the second part, first note that $Q(z)$ has no zero in the open disk $|z| < R$, since otherwise the radius of convergence R of $f(z) = 1/Q(z)$ would be strictly smaller than R. Suppose that $Q(z)$ has no zero on the circle $|z| = R$. Then it has no zero in the open disk $D(0, R + \varepsilon)$ for some $\varepsilon > 0$ (recall that $R < R_1$). It follows that the radius of convergence R of $f(z) = 1/Q(z)$ is strictly greater than R, which is a contradiction. Therefore, we have $R = \min\{|\lambda|; Q(\lambda) = 0\}$ and we have $Q(Re^{i\theta}) = 0$ for some $\theta \in [0, 2\pi)$. □

As an immediate corollary, we have the following.

Corollary 3.3: *In the above theorem, if*

$$\Lambda = \lim_{n\to\infty} \left|\frac{V_n}{V_{n+1}}\right|$$

exists and $\Lambda < R_1$, *then* Λ *is the smallest among the moduli of the zeros of* Q.

Remark 3.4: Even if we assume that $V_n \neq 0$ for all n, we do not have

$$R = \liminf_{n\to\infty} \left|\frac{V_n}{V_{n+1}}\right| \text{ or } R = \limsup_{n\to\infty} \left|\frac{V_n}{V_{n+1}}\right|$$

in general. For example, set

$$g(z) = 1 + z^2 + z^4 + z^6 + \cdots,$$
$$h(z) = z + z(2z)^2 + z(2z)^4 + \cdots = zg(2z),$$

and

$$f(z) = g(z) + h(z) = \sum_{n=0}^{\infty} V_n z^n.$$

The radius of convergence of g is equal to 1, while that of h is equal to 1/2. Hence the radius of convergence of f is equal to $R = 1/2$. However, we have

$$\frac{V_n}{V_{n+1}} = \begin{cases} 2^{-n}, & \text{if } n \text{ is even,} \\ 2^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus we have

$$\limsup_{n\to\infty}\left|\frac{V_n}{V_{n+1}}\right| = +\infty, \quad \liminf_{n\to\infty}\left|\frac{V_n}{V_{n+1}}\right| = 0.$$

So, neither of them gives R in this example.

Remark 3.5: Suppose that

$$\lambda = \lim_{n\to\infty}\frac{V_n}{V_{n+1}}$$

exists. Then we do not know if λ itself is a zero of Q with the smallest modulus. Compare this with Remark 2.4. In §4 we will give a partial answer to this question.

Remark 3.6: In Corollary 3.3, if $\Lambda \geq R_1$, then Q does not have a zero in the open disk $D(0; R_1)$.

By using Proposition 2.7, we can prove the following combinatorial expression for $\{V_n\}_{n\in\boldsymbol{Z}}$.

Proposition 3.7: *Let $\{V_n\}_{n\in\boldsymbol{Z}}$ be a sequence as in* (1.2) *whose coefficients and initial values are $\{a_j\}_{j\geq 0}$ and $V_0 = 1,\ V_{-j} = 0$ for all $j \geq 1$, respectively. Then we have*

$$V_n = \rho(n,n) = \sum_{k_0+2k_1+\cdots+nk_{n-1}=n}\frac{(k_0+k_1+\cdots+k_{n-1})!}{k_0!k_1!\ldots k_{n-1}!}a_0^{k_0}a_1^{k_1}\ldots a_{n-1}^{k_{n-1}} \tag{3.3}$$

for all $n \in \boldsymbol{Z}$.

By Corollary 3.3, if

$$\Lambda = \lim_{n\to\infty}\left|\frac{\rho(n,n)}{\rho(n+1,n+1)}\right|$$

exists and is strictly smaller than R_1, then Λ is the smallest among the moduli of the zeros of Q. Furthermore, the radius of convergence R of the Taylor series (3.2) of $f(z) = 1/Q(z)$ coincides with Λ. We also have $f^{(n)}(0) = n!V_n^{(r)}$ for all $n \geq 0$ as in Remark 2.8.

4. THE BERNOULLI-EULER METHOD BY APPROXIMATION PROCESS

In this section, we will use the results of §2 in order to approximate a zero of a *holomorphic function* by using r-generalized Fibonacci sequences with r finite. The idea is very similar to that of [3].

Let $Q(z)$ be a complex function which is holomorphic in a neighbourhood of the origin. Let $R_1 > 0$ be the largest positive real number such that Q is holomorphic in the open disk $D(0; R_1)$. As in the previous section, we suppose that its Taylor series expansion takes the form (3.1).

Let $\{V_n\}_{n\in\boldsymbol{Z}}$ be an ∞-generalized Fibonacci sequence as in (1.2) whose coefficients and initial values are $\{a_j\}_{j\geq 0}$ and $V_0 = 1,\ V_{-j} = 0$ for all $j \geq 1$, respectively. Note that V_n exists for all $n \in \boldsymbol{Z}$. The following approximation has been established in [3]:

$$V_n = \lim_{r\to\infty} V_n^{(r)} \tag{4.1}$$

for all $n \geq 1$, where for each $r \geq 1$, the sequence $\{V_n^{(r)}\}_{n \geq -r+1}$ is a type (1.1) defined by $V_0^{(r)} = 1$, $V_n^{(r)} = 0$ for $-r+1 \leq n \leq -1$, and $V_{n+1}^{(r)} = a_0 V_n^{(r)} + \cdots + a_{r-1} V_{n-r+1}^{(r)}$ for $n \geq 0$. However, in our case, (4.1) is trivial, since we have $V_n^{(r)} = V_n$ for $r \geq n$.

Our first result of this section is the following.

Theorem 4.1: *Let*

$$Q(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}$$

be a holomorphic function defined in a neighbourhood of the origin with radius of convergence $R_1 > 0$. Consider the doubly indexed sequence $\{V_n^{(r)}\}_{n \geq -r+1, r \geq 1}$ as above. We suppose the following.

(1) $V_n^{(r)} \neq 0$ *for all sufficiently large n and r.*
(2) *For all sufficiently large r,*

$$\lambda^{(r)} = \lim_{n \to \infty} \frac{V_n^{(r)}}{V_{n+1}^{(r)}}$$

exists.
(3) $\lambda = \lim_{r \to \infty} \lambda^{(r)}$ *exists and we have $|\lambda| < R_1$.*

Then λ is a zero of Q.

Proof: Set $Q_r(z) = 1 - a_0 z - \cdots - a_{r-2} z^{r-1} - a_{r-1} z^r$. By Remark 2.4, we have

$$\lim_{n \to \infty} Q_r\left(\frac{V_n^{(r)}}{V_{n+1}^{(r)}}\right) = Q_r\left(\lim_{n \to \infty} \frac{V_n^{(r)}}{V_{n+1}^{(r)}}\right) = Q_r(\lambda^{(r)}) = 0$$

for all sufficiently large r. Set $T_r(z) = Q(z) - Q_r(z)$. Note that for every R_1' with $0 < R_1' < R_1$, we have

$$\lim_{r \to \infty} T_r(z) = 0$$

uniformly for $|z| \leq R_1'$. We have

$$Q(\lambda^{(r)}) = \lim_{n \to \infty} Q\left(\frac{V_n^{(r)}}{V_{n+1}^{(r)}}\right) = \lim_{n \to \infty} T_r\left(\frac{V_n^{(r)}}{V_{n+1}^{(r)}}\right) = T_r(\lambda^{(r)})$$

for all sufficiently large r. Hence we have

$$Q(\lambda) = \lim_{r \to \infty} Q(\lambda^{(r)}) = \lim_{r \to \infty} T_r(\lambda^{(r)}) = 0.$$

This completes the proof. □

As a corollary, we have the following.

Corollary 4.2: *Let*

$$Q(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}$$

be a holomorphic function defined in a neighbourhood of the origin with radius of convergence $R_1 > 0$. *Consider the doubly indexed sequence* $\{V_n^{(r)}\}_{n \geq -r+1, r \geq 1}$ *and the sequence* $\{V_n\}_{n \in \boldsymbol{Z}}$ *as above. We suppose the following.*

(1) $V_n^{(r)}, V_n \neq 0$ *for all sufficiently large* n *and* r.
(2) *For all sufficiently large* r,

$$\lambda^{(r)} = \lim_{n \to \infty} \frac{V_n^{(r)}}{V_{n+1}^{(r)}}$$

exists and converges uniformly with respect to r.
(3) $\lambda = \lim_{r \to \infty} \lambda^{(r)}$ *exists and we have* $|\lambda| < R_1$.
Then we have

$$\lambda = \lim_{n \to \infty} \frac{V_n}{V_{n+1}}$$

and it is a zero of Q.

Proof: By our assumptions, we see that

$$\lim_{n, r \to \infty} \frac{V_n^{(r)}}{V_{n+1}^{(r)}} = \lambda.$$

Then the result follows from (4.1) together with Theorem 4.1. □

Example 4.3: Let us consider the example in [3, §7]. We shall use the same notation. In this example, since the coefficients a_i are all strictly positive real numbers, we have $V_n^{(r)} \neq 0$ for all $n \geq 0$ and $r \geq 1$. It has been shown that the sequences $\{V_n^{(r)}/q_r^n\}_{n \geq 1}$ are uniformly convergent for $r \geq 1$ and that

$$\lim_{n \to \infty} \frac{V_n^{(r)}}{q_r^n} = 1.$$

Since the sequence $\{q_r\}_{r \geq 1}$ converges to $q > 0$, the sequences $\{V_n^{(r)}/V_{n+1}^{(r)}\}_{n \geq 1}$ are also uniformly convergent and converge to $q_r^{-1} = p_r$ for $r \geq 1$. Furthermore, we have

$$\lim_{r \to \infty} p_r = p$$

and $0 < p < R_1$, where R_1 is the radius of convergence of Q (in [3, §7], R_1 is written as R). Thus all the assumptions of Corollary 4.2 are satisfied and p is a root of Q.

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