CERTAIN CLASSES OF FINITE SUMS THAT INVOLVE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The beautiful identity

$$\sum_{n=1}^{j} F_n^2 = F_j F_{j+1} \tag{1.1}$$

was the inspiration for [2], in which analogous sums involving cubes of Fibonacci numbers were developed. In turn, [2] was the motivation for [5], [6], and [7]. In the present paper, where we restrict ourselves to summands that consist of products of at most two terms (as in (1.1)), our motivation has again been to find sums where the right side has a pleasing form. We have found it profitable to consider non-alternating sums, alternating sums, and sums that alternate according to $(-1)^{\frac{n(n+1)}{2}}$. Fibonacci-related sums of the latter type are almost non-existent in the literature.

During the discovery process we became aware of numerous connections that exist between various groups of sums. So, rather than merely to present a collection of sums, a priority of ours has been to highlight the strong thread of unity that exists. Another priority has been to achieve a balance between elegance and generality, and, to achieve this, experimentation has led us to employ the four sequences that we now define.

We define the sequence $\{W_n\}$, for all integers n, by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b, \tag{1.2}$$

where a, b, and p are assumed to be arbitrary complex numbers with $p(p^2+2)(p^2+4) \neq 0$. Then, since $\Delta = p^2 + 4 \neq 0$, the roots α and β of $x^2 - px - 1 = 0$ are distinct. Hence the Binet form (see [3]) for W_n is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

where $A = b - a\beta$ and $B = b - a\alpha$. The Binet form gives W_n for all integers n.

We define another sequence $\{X_n\}$ by $X_n = W_{n+1} + W_{n-1}$, and, with the use of (1.3), we find that

$$X_n = A\alpha^n + B\beta^n \text{ for all integers } n. \tag{1.4}$$

For $(a, b) = (0, 1)$ write $W_n = U_n$ and $X_n = V_n$. Thus W_n and X_n generalize U_n and V_n, respectively, which in turn generalize F_n and L_n, respectively. Aspects of $\{W_n\}$ and $\{X_n\}$ have been treated, for example, in [1], [4], and [11], and more recently in [8].

In Sections 2, 3, and 4 we present our results, and then conclude by giving a sample proof. Since we have decided upon a limited focus, we do not claim that our results are exhaustive. Indeed, we expect that there is scope for further research along the lines that we set forth.

In each of our sums the lower limit is allowed to vary. Accordingly, in keeping with convention, we always assume the upper limit to be greater than the lower limit, and that either limit may be negative.

2. THE FIRST SET OF SUMS

In this section, and the sections that follow, we systematically consider non-alternating sums, alternating sums, and sums that alternate according to $(-1)^{\frac{n(n+1)}{2}}$. We begin with the following three results.

$$\sum_{n=i}^{j} W_n = \begin{cases} \frac{1}{p}V_{(j-i+1)/2}(W_{(j+i+1)/2} + W_{(j+i-1)/2}) & \text{if } j-i \equiv 1(\text{mod } 4) \\ \frac{1}{p}U_{(j-i+1)/2}(X_{(j+i+1)/2} + X_{(j+i-1)/2}) & \text{if } j-i \equiv 3(\text{mod } 4) \end{cases}, \tag{2.1}$$

$$\sum_{n=i}^{j} (-1)^n W_n = \begin{cases} \frac{(-1)^j}{p}V_{(j-i+1)/2}(W_{(j+i+1)/2} - W_{(j+i-1)/2}) & \text{if } j-i \equiv 1(\text{mod } 4) \\ \frac{(-1)^j}{p}U_{(j-i+1)/2}(X_{(j+i+1)/2} - X_{(j+i-1)/2}) & \text{if } j-i \equiv 3(\text{mod } 4) \end{cases}, \tag{2.2}$$

and

$$\sum_{n=i}^{j} (-1)^{\frac{n(n+1)}{2}} W_n = \begin{cases} (-1)^{\frac{j(j+1)}{2}}W_{(j+i)/2}(U_{(j-i+2)/2} + (-1)^j U_{(j-i)/2}) & \text{if } j-i \equiv 0(\text{mod } 2) \\ (-1)^{\frac{j(j+1)}{2}}U_{(j-i+1)/2}(W_{(j+i+1)/2} + (-1)^j W_{(j+i-1)/2}) & \text{if } j-i \equiv 1(\text{mod } 2) \end{cases}. \tag{2.3}$$

Now suppose that, in the summands of (2.1) and (2.2), W is replaced by X. Then if $j - i \equiv 3 \pmod 4$ we modify the right sides by multiplying with Δ and replacing each occurrence of X by W. It is now clear what (2.1)-(2.3) become when any of U_n, V_n, or X_n is substituted for W_n in the summand.

3. THE SECOND SET OF SUMS

In this section we consider sums similar to (2.1)-(2.3) in which W_n is replaced by W_{2n}.

$$\sum_{n=i}^{j} W_{2n} = \begin{cases} \frac{1}{p}V_{j-i+1}W_{j+i} & \text{if } j-i \equiv 0(\text{mod } 2) \\ \frac{1}{p}U_{j-i+1}X_{j+i} & \text{if } j-i \equiv 1(\text{mod } 2) \end{cases}, \tag{3.1}$$

and

$$\sum_{n=i}^{j} X_{2n} = \frac{\Delta}{p}U_{j-i+1}W_{j+i} \qquad \text{if } j-i \equiv 1 \pmod 2. \tag{3.2}$$

In the alternating sum that follows the parity of the limits is not an issue.

$$\sum_{n=i}^{j} (-1)^n W_{2n} = (-1)^j U_{j-i+1}W_{j+i}. \tag{3.3}$$

For the following group of sums the upper and lower limits are specified as belonging to certain residue classes modulo 4.

$$\sum_{n=4i+1}^{4j}(-1)^{\frac{n(n+1)}{2}}W_{2n}=\frac{\Delta}{\Delta-2}U_{4j-4i}W_{4j+4i+1}, \tag{3.4}$$

$$\sum_{n=4i+3}^{4j}(-1)^{\frac{n(n+1)}{2}}W_{2n}=\frac{1}{\Delta-2}V_{4j-4i-2}X_{4j+4i+3}, \tag{3.5}$$

$$\sum_{n=4i}^{4j+3}(-1)^{\frac{n(n+1)}{2}}W_{2n}=\frac{p}{\Delta-2}U_{4j-4i+4}X_{4j+4i+3}, \tag{3.6}$$

$$\sum_{n=4i+2}^{4j+3}(-1)^{\frac{n(n+1)}{2}}W_{2n}=\frac{p}{\Delta-2}V_{4j-4i+2}W_{4j+4i+5}. \tag{3.7}$$

We now make an observation concerning the subscripts on the right sides of (3.1)-(3.7). The subscripts of U and V are (upper limit-lower limit+1), while the subscripts of W and X are (upper limit + lower limit). The reader can observe that this also applies to most of the sums in this paper.

Notice that, in (3.4)-(3.7), one limit of summation is even while the other is odd. Accordingly, we have observed that each of (3.4)-(3.7) has a **dual** sum that is obtained with the use of the rule below. We highlight this rule since it also applies to certain groups of sums in Section 4.

Rule for the Formation of the Dual Sum

- Replace the even limit by the even limit corresponding to the *other* residue class modulo 4 and the odd limit by the odd limit corresponding to the *other* residue class modulo 4
- Calculate the subscripts on the right in accordance with the paragraph following (3.7)
- Multiply the right side by -1.

For example, the dual of (3.7) is

$$\sum_{n=4i}^{4j+1}(-1)^{\frac{n(n+1)}{2}}W_{2n}=\frac{-p}{\Delta-2}V_{4j-4i+2}W_{4j+4i+1}. \tag{3.8}$$

We also remark that if, in (3.5) and (3.6), W in the summand is replaced by X, then on the right we replace X by W and multiply by Δ. This also applies to the duals of (3.5) and (3.6). It is now clear what the summation identities of this section become when W in the summand is replaced by either U, V, or X.

Finally if, in each sum of this seciton, W_{2n} is replaced by W_{2n+k}, the subscript of W or X on the right side is simply increased by k. The same applies to each sum where X_{2n} occurs in the summand.

4. THE THIRD SET OF SUMS

In this section we consider sums where the summand contains a second order product. Since, for example, $U_nV_n = U_{2n}$, sums that involve the product U_nV_n follow from Section 3. However, sums that involve U_n^2 or V_n^2 do not follow from anything we have done so far. To remedy this, and to achieve more generality, we next consider sums in which the summand contains U_nW_n, U_nX_n, V_nW_n, or V_nX_n.

In the three sums that follow i and j are assumed to have *different* parities. Under this assumption we have

$$\sum_{n=i}^{j} U_nW_n = \frac{1}{p}U_{j-i+1}W_{j+i}, \tag{4.1}$$

$$\sum_{n=i}^{j} V_nW_n = \frac{1}{p}U_{j-i+1}X_{j+i}, \tag{4.2}$$

and

$$\sum_{n=i}^{j} V_nX_n = \frac{\Delta}{p}U_{j-i+1}W_{j+i}, \tag{4.3}$$

It is clear that the corresponding sum involving the summand U_nX_n can be obtained from (4.1).

Next we consider the associated alternating sums, where, unfortunately, the right sides do not factorise nicely. Nevertheless, for the sake of completeness, we have managed to salvage something in the spirit of our previous results. Without any assumptions on the parities of the limits we have

$$\sum_{n=i}^{j}(-1)^nU_nW_n = \frac{1}{\Delta}\left((-1)^jU_{j-i+1}X_{j+i} - (j-i+1)X_0\right), \tag{4.4}$$

and

$$\sum_{n=i}^{j}(-1)^nU_nX_n = (-1)^jU_{j-i+1}W_{j+i} - (j-i+1)W_0. \tag{4.5}$$

In (4.4), if U_nW_n is replaced with V_nX_n, the right side is multiplied by Δ and the sign of the coefficient of X_0 is changed. In (4.5), if U_nX_n is replaced by V_nW_n, then only the sign of the coefficient of W_0 is changed on the right side.

The next group of sums has $(-1)^{\frac{n(n+1)}{2}}$ in the summand.

$$\sum_{n=4i+1}^{4j}(-1)^{\frac{n(n+1)}{2}}U_nX_n = \frac{\Delta}{\Delta-2}U_{4j-4i}W_{4j+4i+1}, \tag{4.6}$$

$$\sum_{n=4i+3}^{4j} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \frac{1}{\Delta - 2} V_{4j-4i-2} X_{4j+4i+3}, \tag{4.7}$$

$$\sum_{n=4i}^{4j+3} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \frac{p}{\Delta - 2} U_{4j-4i+4} X_{4j+4i+3}, \tag{4.8}$$

$$\sum_{n=4i+2}^{4j+3} (-1)^{\frac{n(n+1)}{2}} U_n X_n = \frac{p}{\Delta - 2} V_{4j-4i+2} W_{4j+4i+5} + 2W_0. \tag{4.9}$$

The right sides of these four sums should be compared, respectively, with the right sides of (3.4)-(3.7). The only difference occurs in (4.9) where the term $2W_0$ is added. Furthermore, each of (4.6)-(4.9) has a dual sum that is obtained with the use of the rule in Section 3. And from this total of eight sums we obtain a further eight sums if, in each summand, we replace $U_n X_n$ by $V_n W_n$. In each case the right side remains unchanged, except for (4.9) and its dual, where the coefficient of W_0 undergoes a change in sign.

To conclude the list of sums in this paper we list four more, and, as in the previous paragraph, we describe how to obtain an additional twelve. We have found the following.

$$\sum_{n=4i+1}^{4j} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{1}{\Delta - 2} U_{4j-4i} X_{4j+4i+1}, \tag{4.10}$$

$$\sum_{n=4i+3}^{4j} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{1}{\Delta - 2} V_{4j-4i-2} W_{4j+4i+3}, \tag{4.11}$$

$$\sum_{n=4i}^{4j+3} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{p}{\Delta - 2} U_{4j-4i+4} W_{4j+4i+3}, \tag{4.12}$$

$$\sum_{n=4i+2}^{4j+3} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{1}{\Delta} \left(\frac{p}{\Delta - 2} V_{4j-4i+2} X_{4j+4i+5} + 2X_0 \right). \tag{4.13}$$

Each of (4.10)-(4.13) has a dual sum that is obtained with the use of the rule in Section 3. And from this total of eight sums we obtain a further eight sums if, in each summand, we replace $U_n W_n$ by $V_n X_n$ and multiply the right side by Δ.

Multiplication by Δ is the only change we are required to make to the right side, except for (4.13) and its dual, where, in addition, we must change the sign of the coefficient of X_0. For example, the dual of (4.13) is

$$\sum_{n=4i}^{4j+1} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{-1}{\Delta} \left(\frac{p}{\Delta - 2} V_{4j-4i+2} X_{4j+4i+1} + 2X_0 \right), \tag{4.14}$$

and an additional sum is

$$\sum_{n=4i}^{4j+1}(-1)^{\frac{n(n+1)}{2}}V_nX_n = -\left(\frac{p}{\Delta-2}V_{4j-4i+2}X_{4j+4i+1}-2X_0\right). \tag{4.15}$$

By way of example, if we put $W_n = F_n$ then (3.3), (3.5), and (4.12) become, respectively,

$$\sum_{n=i}^{j}(-1)^nF_{2n} = (-1)^jF_{j-i+1}F_{j+1}, \tag{4.16}$$

$$\sum_{n=4i+3}^{4j}(-1)^{\frac{n(n+1)}{2}}F_{2n} = \frac{1}{3}L_{4j-4i-2}L_{4j+4i+3}, \tag{4.17}$$

and

$$\sum_{n=4i}^{4j+3}(-1)^{\frac{n(n+1)}{2}}F_n^2 = \frac{1}{3}F_{4j-4i+4}F_{4j+4i+3}. \tag{4.18}$$

5. A SAMPLE PROOF

Each of our sums can be proved by induction, and we illustrate by proving (4.12). We require the following result, which is a special case of (79) in [1].

$$X_{n+k}-X_{n-k} = \Delta U_kW_n, \quad k \text{ even}. \tag{5.1}$$

By our earlier assumption on the limits, the smallest allowable value of j is i. Thus, we begin our inductive proof by verifying that

$$\sum_{n=4i}^{4i+3}(-1)^{\frac{n(n+1)}{2}}U_nW_n = \frac{p}{p^2+2}U_4W_{8i+3}. \tag{5.2}$$

We recall that $\Delta = p^2+4 = (\alpha-\beta)^2$, replace U_n and W_n with their Binet forms, and expand to obtain

$$\begin{aligned}
\text{LHS} &= \frac{1}{\Delta}((X_{8i+6}-X_{8i+4})-(X_{8i+2}-X_{8i}))\\
&= \frac{p}{\Delta}(X_{8i+5}-X_{8i+1}) \text{ (from the recurrence for } X_n)\\
&= \frac{p}{\Delta}(X_{8i+3+2}-X_{8i+3-2})\\
&= p^2W_{8i+3} \quad \text{(from (5.1))}
\end{aligned}$$

which is the right side of (5.2) since $U_4 = p^3 + 2p$.

Next, if (4.12) is true for the parameter j, then

$$\sum_{n=4i}^{4(j+1)+3} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{p}{p^2+2} U_{4j-4i+4} W_{4j+4i+3} + \sum_{n=4j+4}^{4j+7} (-1)^{\frac{n(n+1)}{2}} U_n W_n, \tag{5.3}$$

and we are required to prove that the right side of (5.3) is equal to $\frac{p}{p^2+2} U_{4j-4i+8} W_{4j+4i+7}$. Equivalently, we are required to prove that

$$\sum_{n=4j+4}^{4j+7} (-1)^{\frac{n(n+1)}{2}} U_n W_n = \frac{p}{p^2+2} (U_{4j-4i+8} W_{4j+4i+7} - U_{4j-4i+4} W_{4j+4i+3}).$$

Proceeding as before, we find

$$\begin{aligned} \text{LHS} &= \frac{1}{\Delta}(X_{8j+14} - X_{8j+12} - (X_{8j+10} - X_{8j+8})) \\ &= \frac{p}{\Delta}(X_{8j+13} - X_{8j+9}) \\ &= p^2 W_{8j+11} \quad \text{(from (5.1)). Similarly} \end{aligned}$$

$$\text{RHS} = \frac{p}{(p^2+2)\Delta}(X_{8j+15} - X_{8j+7}) = \frac{p}{(p^2+2)\Delta} \times \frac{\Delta U_4 W_{8j+11}}{1} = \text{ LHS},$$

and the proof is complete.

6. CONCLUDING COMMENTS

In [9] Russell considers the sums $\sum_{n=i}^{j} R_n$ and $\sum_{n=i}^{j} R_n S_n$ with no restrictions on the limits of summation. Here $\{R_n\}$ and $\{S_n\}$ are sequences generated by the recurrence $W_n = pW_{n-1} + qW_{n-2}$ with p and q real. In each of these sums several cases are given depending on the values of p and q. Indeed, there are three cases for the first sum and seven cases for the second. The character of Russell's sums and ours is quite different, since our motivation has been to present sums in which the right side has a pleasing form. In a subsequent paper, Russell [10] gives finite sums in which each summand consists of products of up to three terms, where each term is generated by $W_n = W_{n-1} + W_{n-2}$. Neither of Russell's papers contains alternating sums or sums that alternate according to $(-1)^{\frac{n(n+1)}{2}}$. However, they are the only papers we have seen that contain finite sums where each summand consists of terms (or products of terms) generated by second-order linear recurrences, and where the lower limit of summation is allowed to vary.

We discovered each of our results numerically by first considering the Fibonacci and Lucas sequences. Initially all lower limits were taken to be one, and in each case we varied the upper limit until the sum could be expressed as a product of Fibonacci and/or Lucas numbers. We then decided to vary the upper and lower limits simultaneously and then found that our results could be translated to the more general sequences defined herein. The process was quite tedious, and we gratefully acknowledge our use of the computer algebra package *Mathematica* 3.0.

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