SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS AND LUCAS NUMBERS

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(Submitted August 2001–Final Revision March 2003)

1. INTRODUCTION AND RESULTS

As usual, the Fibonacci sequence $\{F_n\}$ and the Lucas sequences $\{L_n\}(n = 0, 1, 2, \dots,)$ are defined by the second-order linear recurrence sequences

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n$$

for $n \geq 0, F_0 = 0, F_1 = 1, L_0 = 2$ and $L_1 = 1$. These sequences play a very important role in the studied of the theory and application of mathematics. Therefore, the various properties of F_n and L_n were investigated by many authors. For example, R. L. Duncan [2] and L. Kuipers [5] proved that $(\log F_n)$ is uniformly distributed mod 1. Neville Robbins [4] studied the Fibonacci numbers of the forms $px^2 \pm 1, px^3 \pm 1$, where p is a prime. The author [6] and Fengzhen Zhao [3] obtained some identities involving the Fibonacci numbers. In this paper, as a generalization of [3] and [6], we shall use elementary methods to study the calculating problems of the general summations

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{m(a_1+1)} \cdot F_{m(a_2+1)} \dots F_{m(a_k+1)} \tag{1}$$

and

$$\sum_{a_1+a_2+\cdots+a_k=n} L_{ma_1} \cdot L_{ma_2} \dots L_{ma_k}, \tag{2}$$

and give two exact calculating formulas, where the summation is taken over all k-dimension nonnegative integer coordinates $(a_1, a_2, \dots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n, k$ and m are any positive integers, and n be any nonnegative integer.

For convenience, we first define Chebyshev polynomials of the first and second kind $T(x) = \{T_n(x)\}$ and $U(x) = \{U_n(x)\}(n = 0, 1, 2, \dots,)$ as follows:

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x) \tag{3}$$

and

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) \tag{4}$$

for $n \geq 0, T_0(x) = 1, T_1(x) = x, U_0(x) = 1$ and $U_1(x) = 2x$. Let $U_n^{(k)}(x)$ denote the k^{th} derivative of $U_n(x)$ with respect to x. We will use generating functions for the sequences $T_n(x)$ and $U_n(x)$ and their partial derivatives to prove the following two theorems.

Theorem 1: *For any positive integer k, m and nonnegative integer n, we have the identity*

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{m(a_1+1)} \cdot F_{m(a_2+1)} \dots F_{m(a_{k+1}+1)} = (-i)^{mn} \frac{F_m^{k+1}}{2^k \cdot k!} U_{n+k}^{(k)}\left(\frac{i^m}{2} L_m\right),$$

where i is the square root of -1.

Theorem 2: *For any positive integer k, m and nonnegative integer n, we have*

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n+k+1} L_{ma_1}\cdot L_{ma_2}\dots L_{ma_{k+1}}$$

$$= (-i)^{m(n+k+1)}\frac{2}{k!}\sum_{h=0}^{k+1}\left(\frac{i^{m+2}}{2}L_m\right)^h\binom{k+1}{h}U^{(k)}_{n+2k+1-h}\left(\frac{i^m}{2}L_m\right),$$

where $\binom{k+1}{h}=\frac{(k+1)!}{h!\cdot(k+1-h)!}$.

¿From these two theorems we may immediately deduce the following corollaries:

Corollary 1: *For any positive integer m and nonnegative integer n, we have the identities*

$$\sum_{a_1+a_2+a_3=n} F_{m(a_1+1)}\cdot F_{m(a_2+1)}\cdot F_{m(a_3+1)} = \frac{3}{2}\frac{(-1)^{m-1}F_m^2}{4-(-1)^mL_m^2}\times$$

$$\left[\frac{(n+2)(n+4)}{3}F_{m(n+3)}-\frac{2(n+3)L_m}{4-(-1)^mL_m^2}F_{m(n+2)}+\frac{(n+2)(-1)^mL_m^2}{4-(-1)^mL_m^2}F_{m(n+3)}\right].$$

In particular, for $m = 2, 3, 4$ and 5, we have the identities

$$\sum_{a_1+a_2+a_3=n} F_{2(a_1+1)}\cdot F_{2(a_2+1)}\cdot F_{2(a_3+1)} = \frac{1}{50}[18(n+3)F_{2n+4}+(n+2)(5n-7)F_{2n+6}],$$

$$\sum_{a_1+a_2+a_3=n} F_{3(a_1+1)}\cdot F_{3(a_2+1)}\cdot F_{3(a_3+1)} = \frac{1}{50}[(n+2)(5n+8)F_{3n+9}-6(n+3)F_{3n+6}],$$

$$\sum_{a_1+a_2+a_3=n} F_{4(a_1+1)}\cdot F_{4(a_2+1)}\cdot F_{4(a_3+1)} = \frac{1}{150}[(n+2)(15n+11)F_{4(n+3)}+14(n+3)F_{4(n+2)}]$$

and

$$\sum_{a_1+a_2+a_3=n} F_{5(a_1+1)}\cdot F_{5(a_2+1)}\cdot F_{5(a_3+1)} = \frac{1}{1250}[(n+2)(125n+137)F_{5(n+3)}-66(n+3)F_{5(n+2)}].$$

Corollary 2: *For any positive integer k and nonnegative integer n, we have the identities*

$$\sum_{a_1+a_2+a_3=n+3} L_{a_1}\cdot L_{a_2}\cdot L_{a_3} = \frac{n+5}{2}[(n+10)F_{n+3}+2(n+7)F_{n+2}],$$

$$\sum_{a_1+a_2+a_3=n+3} L_{2a_1}\cdot L_{2a_2}\cdot L_{2a_3} = \frac{n+5}{2}[3(n+10)F_{2n+5}+(n+16)F_{2n+4}]$$

and

$$\sum_{a_1+a_2+a_3=n+3} L_{3a_1} \cdot L_{3a_2} \cdot L_{3a_3} = \frac{n+5}{2}[4(n+10)F_{3n+7} + 3(n+9)F_{3n+6}].$$

Corollary 3: *For any positive integer m and nonnegative integer n, we have the congruences*

$$(n+2)(4n+16-(-1)^m L_m^2) \cdot F_{m(n+3)} \equiv 6(n+3) \cdot L_m \cdot F_{m(n+2)} \bmod 2(4-(-1)^m L_m^2)^2 \cdot F_m.$$

In particular, for $m = 3, 4$ and 5, we have

$$(n+2)(5n+8)F_{3n+9} \equiv 6(n+3)F_{3n+6} \bmod 400;$$

$$(n+2)(15n+11)F_{4(n+3)} + 14(n+3)F_{4(n+2)} \equiv 0 \bmod 4050;$$

$$(n+2)(125n+137)F_{5(n+3)} \equiv 66(n+3)F_{5(n+2)} \bmod 156250.$$

2. SEVERAL LEMMAS

In this section, we shall give several lemmas which are necessary in the proofs of the theorems. First we need two exact expressions and generating functions on $T_n(x)$ and $U_n(x)$ (see (2.1.1) of [1]). That is,

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right] \tag{5}$$

and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right]. \tag{6}$$

So we can easily deduce that the generating function of $T(x)$ nd $U(x)$ are

$$G(t,x) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{+\infty} T_n(x) \cdot t^n \tag{7}$$

and

$$F(t,x) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{+\infty} U_n(x) \cdot t^n. \tag{8}$$

Applying these generating functions we can easily deduce the following

Lemma 1: *For any positive integer k and nonnegative integer n, we have the identity*

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x) \cdot U_{a_2}(x) \dots U_{a_{k+1}}(x) = \frac{1}{2^k \cdot k!} U_{n+k}^{(k)}(x).$$

Proof: Differentiating (8) we obtain

$$\frac{\partial F(t,x)}{\partial x} = \frac{2t}{(1-2xt+t^2)^2} = \sum_{n=0}^{\infty} U_{n+1}^{(1)}(x) \cdot t^{n+1};$$

$$\frac{\partial_2 F(t,x)}{\partial x^2} = \frac{2! \cdot (2t)^2}{(1-2xt+t^2)^3} = \sum_{n=0}^{\infty} U_{n+2}^{(2)}(x) \cdot t^{n+2};$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\partial^k F(t,x)}{\partial x^k} = \frac{k! \cdot (2t)^k}{(1-2xt+t^2)^{k+1}} = \sum_{n=0}^{\infty} U_{n+k}^{(k)}(x) \cdot t^{n+k}. \tag{9}$$

where we have used the fact that $U_n(x)$ is a polynomial of degree n.

Therefore, from (9) we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\ldots U_{a_{k+1}}(x)\right)\cdot t^n = \left(\sum_{n=0}^{\infty} U_n(x)\cdot t^n\right)^{k+1}$$

$$= \frac{1}{(1-2xt+t^2)^{k+1}} = \frac{1}{k!(2t)^k}\frac{\partial^k F(t,x)}{\partial x^k} = \frac{1}{2^k \cdot k!}\sum_{n=0}^{\infty} U_{n+k}^{(k)}(x)\cdot t^n. \tag{10}$$

Equating the coefficients of t^n on both sides of equation (10) we obtain the identity

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\ldots U_{a_{k+1}}(x) = \frac{1}{2^k \cdot k!}\cdot U_{n+k}^{(k)}(x).$$

This proves Lemma 1.

Lemma 2: *For any positive integer k and nonnegative integer n, we have*

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n+k+1} T_{a_1}(x)\cdots T_{a_{k+1}}(x) = \frac{1}{2^k \cdot k!}\sum_{h=0}^{k+1}(-x)^h\binom{k+1}{h}U_{n+2k+1-h}^{(k)}(x).$$

Proof: To prove Lemma 2, multiplying $(1-xt)^{k+1}$ on both sides of (9) we have

$$\frac{(1-xt)^{k+1}}{(1-2xt+t^2)^{k+1}} = \frac{1}{2^k \cdot k!}\sum_{n=0}^{\infty} U_{n+k}^{(k)}(x)\cdot t^n(1-xt)^{k+1}. \tag{11}$$

Note that $(1-xt)^{k+1} = \sum_{h=0}^{k+1}(-x)^h t^h\binom{k+1}{h}$. Comparing the coefficients of t^{n+k+1} on both sides of equation (11) we obtain Lemma 2.

Lemma 3: *For any positive integers m and n, we have the identities*

$$T_n(T_m(x)) = T_{mn}(x) \quad \text{and} \quad U_n(T_m(x)) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)}.$$

Proof: For any positive integer m, from (5) we have

$$T_m^2(x) - 1 = \frac{1}{4}\left[(x+\sqrt{x^2-1})^m + (x-\sqrt{x^2-1})^m\right]^2 - 1$$

$$= \frac{1}{4}\left[(x+\sqrt{x^2-1})^m - (x-\sqrt{x^2-1})^m\right]^2$$

or

$$\sqrt{T_m^2(x)-1} = \frac{1}{2}\left[(x+\sqrt{x^2-1})^m - (x-\sqrt{x^2-1})^m\right].$$

Thus,

$$T_m(x) + \sqrt{T_m^2(x)-1} = (x+\sqrt{x^2-1})^m. \tag{12}$$

$$T_m(x) - \sqrt{T_m^2(x)-1} = (x-\sqrt{x^2-1})^m. \tag{13}$$

Combining (6), (12) and (13) we have

$$U_n(T_m(x)) = \frac{1}{2\sqrt{T_m^2(x)-1}}\left[\left(T_m(x)+\sqrt{T_m^2(x)-1}\right)^{n+1} - \left(T_m(x)-\sqrt{T_m^2(x)-1}\right)^{n+1}\right]$$

$$= \frac{(x+\sqrt{x^2-1})^{m(n+1)} - (x-\sqrt{x^2-1})^{m(n+1)}}{(x+\sqrt{x^2-1})^m - (x-\sqrt{x^2-1})^m}$$

$$= \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)}.$$

Similarly, we can also deduce that $T_n(T_m(x)) = T_{mn}(x)$. This proves Lemma 3.

3. PROOF OF THE THEOREMS

Now we complete the proofs of the theorems. Let i be the square root of -1. Taking $x = T_m(\frac{i}{2})$ in Lemma 1 and Lemma 2, and noting that $U_n(\frac{i}{2}) = i^n F_{n+1}, T_n(\frac{i}{2}) = \frac{i^n}{2} L_n, T_n(T_m(\frac{i}{2})) = \frac{i^{mn}}{2} L_{mn}, U_n(T_m(\frac{i}{2})) = i^{mn}\frac{F_{m(n+1)}}{F_m}$, we may immediately deduce Theorem 1 and Theorem 2.

Proof of the Corollaries: First we note that $U_n(x)$ satisfies the differential equations

$$(1-x^2)U_n'(x) = (n+1)U_{n-1}(x) - nxU_n(x) \tag{14}$$

and

$$(1-x^2)U_n''(x) = 3xU_n'(x) - n(n+2)U_n(x), \tag{15}$$

So from Lemma 3, (14) and (15) we obtain

$$U_n'\left(T_m\left(\frac{i}{2}\right)\right) = \frac{4}{4-(-1)^m L_m^2}\left[i^{m(n-1)}\frac{(n+1)F_{mn}}{F_m} - i^{m(n+1)}\frac{nL_mF_{m(n+1)}}{2F_m}\right]$$

and

$$U_n''\left(T_m\left(\frac{i}{2}\right)\right) = \frac{4i^{mn}}{F_m(4-(-1)^m L_m^2)}$$
$$\times\left[\frac{6(n+1)L_m}{4-(-1)^m L_m^2}F_{mn} - \frac{(-1)^m 3nL_m^2}{4-(-1)^m L_m^2}F_{m(n+1)} - n(n+2)F_{m(n+1)}\right]. \tag{16}$$

Now Corollary 1 and Corollary 2 follows from the recurrence formula

$$F_{n+2} = F_{n+1} + F_n,$$

(16), Theorem 1 and Theorem 2 (with $k=2$).

Corollary 3 follows from Corollary 1 and the fact that $F_m | F_{m(a+1)}$ for all integer $a \geq 0$.

ACKNOWLEDGMENTS

The author expresses his gratitude to the referee for his very helpful and detailed comments.

This work is supported by the N.S.F. (10271093) and P.N.S.F. (2002A11) of P.R. China.

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AMS Classification Numbers: 11B37, 11B39

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