ON DEDEKIND SUMS AND LINEAR RECURRENCES OF ORDER TWO

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INTRODUCTION

If h and k are integers such that $(h,k)=1$ and $k>0$, the Dedekind sum $s(h,k)$ is defined by

$$s(h,k)=\sum_{r=1}^{k}\frac{r}{k}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]-\frac{1}{2}\right)$$

(See Apostol [1], p. 61). Theorem 0 below is Exercise 12a, p. 72 of [1].

Theorem 0: $s(F_{2n},F_{2n+1})=0$ for all $n\geq 1$.

In this note, we (1) generalize Theorem 0 to somewhat more general linear recurrences of order two, and (2) obtain a theorem concerning Lucas numbers that is analogous to Theorem 0.

PRELIMINARIES

$$\text{If } h'\equiv \pm h \pmod{k}, \text{ then } s(h',k)=\pm s(h,k) \tag{1}$$

$$\text{If } h^2+1\equiv 0 \pmod{k}, \text{ then } s(h,k)=0 \tag{2}$$

$$s(h,k)+s(k,h)=\frac{h^2+k^2+1-3hk}{12hk} \tag{3}$$

Let P,Q be integers such that $(P,Q)=1$ and $D=P^2+4Q\neq 0$. Let $\alpha=\frac{P+\sqrt{D}}{2},\beta=\frac{P-\sqrt{D}}{2}$, so that $\alpha-\beta=P,\alpha\beta=-Q$. Let two linear recurrences of order 2 be defined for $n\geq 0$ by:

$$u_0=0,u_1=1,u_n=Pu_{n-1}+Qu_{n-2} \text{ for } n\geq 2 \tag{4}$$

$$v_0=2,v_1=P,v_n=Pv_{n-1}+Qv_{n-2} \text{ for } n\geq 2 \tag{5}$$

(In particular, if $P=Q=1$, then $u_n=F_n$ and $v_n=L_n$.)

The Binet equation state that:

$$u_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},v_n=\alpha^n+\beta^n. \tag{6}$$

Some consequences of (6) are:

$$u_{2n+1}u_{2n-1}-u_{2n}^2=Q^{2n-1} \tag{7}$$

$$v_{2n}^2-v_{2n+1}v_{2n-1}=DQ^{2n-1} \tag{8}$$

$$F_{2n}L_{2n-1} - F_{2n-2}L_{2n+1} = 1. \tag{9}$$

Remarks: (1), (2), (3) are well-known properties of Dedekind sums. (See [1], p. 62.) (4) through (8) are well-known properties of linear recurrences of order 2. (See [2], p. 193-194). (9) can be proved via (6) or by induction on n.

THE MAIN RESULTS

Theorem 1: Let $\{u_n\}$ be a linear recurrence of order 2 (as in (4) above) with $Q = 1$. Then $s(u_{2n}, u_{2n+1}) = 0$.

Proof: Applying (7) and the hypothesis, we get $u_{2n}^2 \equiv -1 \pmod{u_{2n+1}}$. The conclusion now follows from (2).

Theorem 2:

$$s(L_{2n}, L_{2n+1}) = -\frac{F_{2n}}{L_{2n+1}}$$

Proof: Since $L_{2n} = L_{2n+1} - L_{2n-1}$, it suffices (by (1)) to prove that

$$s(L_{2n-1}, L_{2n+1}) = \frac{F_{2n}}{L_{2n+1}}$$

We use induction on n. The theorem holds for $n = 1$, since

$$s(L_1, L_3) = s(1, 4) = \frac{1}{8} = \frac{F_2}{2L_3}$$

(3) implies

$$s(L_{2n-1}, L_{2n+1}) + s(L_{2n+1}, L_{2n-1}) = \frac{L_{2n+1}^2 2 + L_{2n-1}^2 - 3L_{2n+1}L_{2n-1} + 1}{12L_{2n+1}L_{2n-1}}$$

(1) and (5) imply

$$s(L_{2n+1}, L_{2n-1}) = s(L_{2n-2}, L_{2n-1}) = -s(L_{2n-3}, L_{2n-1}) = \frac{F_{2n-2}}{2L_{2n-1}}$$

by induction hypothesis. Furthermore, by (5) and (8), we have

$$L_{2n+1}^2 + L_{2n-1}^2 - 3L_{2n+1}L_{2n-1} = L_{2n}^2 - L_{2n-1}L_{2n+1} = 5.$$

Therefore it suffices to show that

$$\frac{1}{2L_{2n-1}L_{2n+1}} + \frac{F_{2n-2}}{2L_{2n-1}L_{2n+1}} = \frac{F_{2n}}{2L_{2n+1}}.$$

But the last identity follows from (9), so we are done.

REFERENCES

[1] T. Apostol. *Moduluar Functions and Dirichlet Series in Number Theory.* 2nd Ed. (1989), Springer-Verlag.

[2] N. Robbins. *Beginning Number Theory.* (1993) Wm. C. Brown Publishers, Dubuque, IA.

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