ON SOME IDENTITIES INVOLVING THE CHEBYSHEV POLYNOMIALS

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(Submitted November 2001- Final Revision November 2002)

1. INTRODUCTION

In [3], Melham considered sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ defined by

$$U_n = pU_{n-1} - U_{n-2}, \quad U_0 = 0, \quad U_1 = 1,$$

$$V_n = pV_{n-1} - V_{n-2}, \quad V_0 = 2, \quad V_1 = p,$$

where $p \geq 2$. If $p = 2$, then $U_n = n$ and $V_n = 2$ for all $n \geq 0$. For $p > 2$, if α and β, assumed distinct, are the roots of $x^2 - px + 1 = 0$, the Binet's formula are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \textit{ and } V_n = \alpha^n + \beta^n.$$

It was remarked in [2] by Grabner and Prodinger that up to simple changes of variable these polynomials are Chebyshev polynomials, that is

$$U_n(p) = \mathcal{U}_{n-1}\left(\frac{p}{2}\right),$$
$$V_n(p) = 2\mathcal{T}_n\left(\frac{p}{2}\right)$$

where $\mathcal{T}_n$ and $\mathcal{U}_n$ denote the classical Chebyshev polynomials of the first and second kind, respectively. Throughout this paper, let s be an arbitrary positive integer. Let $W_n(a,b) = aU_n + bV_n$ and

$$W_n^{2k}(a,b) + W_{n+s}^{2k}(a,b) = \sum_{r=0}^{k} A_r(a,b;k,s)W_n^{k-r}(a,b)W_{n+s}^{k-r}(a,b). \tag{1}$$

Melham [3] conjectured that $A_r(1,0;k,1) = \frac{\mathcal{D}^r V_k}{r!}$, where $\mathcal{D}$ means differentiation with respect to p. This conjecture was proved in [2] by Grabner and Prodinger in a more general setting that contains Melham's conjecture as a special case. Let $\Omega = a^2 + 4b^2 - b^2p^2$. Grabner and Prodinger obtained that

$$A_r(a,b;k,1) = \Omega^r \sum_{0 \leq 2j \leq k-r} (-1)^j \frac{k(k-1-j)!}{r!j!(k-r-2j)!} p^{k-r-2j}$$

and $A_0(a,b;0,1) = 2$. Furthermore, Grabner and Prodinger also obtained that

$$A_r(a,b;k,2) = \Omega^r \sum_{0 \leq \lambda \leq k-r}$$

$$(-1)^{\lambda} p^{2k-2\lambda} \frac{k\left(k - \lfloor\frac{\lambda}{2}\rfloor - 1\right)! 2^{\lceil\frac{\lambda}{2}\rceil}}{r!\lambda!(k-r-\lambda)!} \prod_{i=0}^{\lfloor\frac{\lambda}{2}\rfloor - 1} \left(2k - 2\lceil\frac{\lambda}{2}\rceil - 1 - 2i\right)$$

and $A_0(a,b;0,2)=2$.

In this note, we obtain some identities involving the Chebyshev polynomials. This generalizes Melham's and Grabner and Prodinger's results.

2. MAIN THEOREMS AND THEIR PROOFS

Lemma 2.1:

$$A_r(a,b;k+1,s)=V_sA_r(a,b;k,s)+\Omega U_s^2A_{r-1}(a,b;k,s)-A_r(a,b;k-1,s); \tag{2}$$

$$A_r(a,b;k,s)=0,\ for\ r>k\ or\ r<0\ or\ k<0; \tag{3}$$

$$A_0(a,b;0,s)=2,\ A_0(a,b;1,s)=V_s,\ A_1(a,b;1,s)=\Omega U_s^2; \tag{4}$$

$$A_0(a,b;k,s)=V_{ks}\ (k\geq 0). \tag{5}$$

Proof: Obviously, (3) and $A_0(a,b;0,s)=2$ hold. Using the Binet's formula of U_n and V_n we have

$$W_n^2(a,b)+W_{n+s}^2(a,b)=V_sW_n(a,b)W_{n+s}(a,b)+\Omega U_s^2. \tag{6}$$

¿From (6), $A_0(a,b;1,s)=V_s$ and $A_1(a,b;1,s)=\Omega_s$ hold immediately. Noting that

$$W_n^{2(k+1)}(a,b)+W_{n+s}^{2(k+1)}(a,b)=\left(W_n^2(a,b)+W_{n+s}^2(a,b)\right)\left(W_n^{2k}(a,b)+W_{n+s}^{2k}(a,b)\right)$$

$$-W_n^2(a,b)W_{n+s}^2(a,b)\left(W_n^{2(k-1)}(a,b)+W_{n+s}^{2(k-1)}(a,b)\right)$$

and applying (6) we have

$$\sum_{r=0}^{k+1}A_r(a,b;k+1,s)W_n^{k+1-r}(a,b)W_{n+s}^{k+1-r}(a,b)$$

$$=\left(V_sW_n(a,b)W_{n+s}(a,b)+\Omega U_s^2\right)\left(\sum_{r=0}^{k}A_r(a,b;k,s)W_n^{k-r}(a,b)W_{n+s}^{k-r}(a,b)\right)$$

$$-W_n^2(a,b)W_{n+s}^2(a,b)\sum_{r=0}^{k-1}A_r(a,b;k-1,s)W_n^{k-1-r}(a,b)W_{n+s}^{k-1-r}(a,b).$$

Comparing the coefficients of $W_n^{k+1-r}(a,b)W_{n+s}^{k+1-r}(a,b)$ yields (2) and $A_0(a,b;k+1,s)=V_sA_0(a,b;k,s)-A_0(a,b;k-1,s)$. Solving this recurrence relation we obtain $A_0(a,b;k,s)=V_{ks}$.

Theorem 2.2:

$$A_r(a,b;k,s)=\Omega^rU_s^{2r}\left(\left[x^{k-r}\right]\frac{1}{(1-V_sx+x^2)^{r+1}}-\left[x^{k-r-2}\right]\frac{1}{(1-V_sx+x^2)^{r+1}}\right), \tag{7}$$

where $[x^k]f(x)$ *denotes the coefficient of* x^k *in* $f(x)$.

Proof: Let $f(x,y)=\sum_{k\geq 0,r\geq 0}A_r(a,b;k,s)x^ky^r$. Summing Lemma 2.1, we have

$$\sum_{k\geq 1,r\geq 0}A_r(a,b;k,s)x^{k+1}y^r=\sum_{k\geq 1,r\geq 0}A_r(a,b;k,s)x^{k+1}y^r$$
$$+\sum_{k\geq 1,r\geq 1}\Omega U_s^2A_{r-1}(a,b;k,s)x^{k+1}y^r-\sum_{k\geq 1,r\geq 0}A_r(a,b;k-1,s)x^{k+1}y^r,$$

i.e.,

$$\sum_{k\geq 2,r\geq 0}A_r(a,b;k,s)x^ky^r=xV_s\sum_{k\geq 1,r\geq 0}A_r(a,b;k,s)x^ky^r$$
$$+xy\Omega U_s^2\sum_{k\geq 1,r\geq 0}A_r(a,b;k,s)x^ky^r-x^2\sum_{k\geq 0,r\geq 0}A_r(a,b;k,s)x^ky^r,$$

that is

$$f(x,y)-2-x\left(V_s+\Omega U_s^2y\right)=xV_s(f(x,y)-2)+xy\Omega U_s^2(f(x,y)-2)-x^2f(x,y).$$

Hence we have

$$f(x,y)=\frac{2-V_sx-\Omega U_s^2xy}{1-V_sx-\Omega U_s^2xy+x^2}$$
$$=1+\frac{1-x^2}{1-V_sx-\Omega U_s^2xy+x^2}$$
$$=1+\frac{1-x^2}{1-V_sx+x^2}\frac{1}{1-y\frac{\Omega U_s^2x}{1-V_sx+x^2}}.$$

Comparing the coefficient of $y^r(r\geq 1)$, we have

$$\sum_{k\geq 0}A_r(a,b;k,s)x^k=\Omega^rU_s^{2r}\left\{\frac{(2-V_sx)x^r}{(1-V_sx+x^2)^{r+1}}-\frac{x^r}{(1-V_sx+x^2)^r}\right\}$$
$$=\Omega^rU_s^{2r}x^r\frac{1-x^2}{(1-V_sx+x^2)^{r+1}}.$$

So reading off the coefficient of x^k we get

$$A_r(a,b;k,s)=\Omega^rU_s^{2r}\left(\left[x^{k-r}\right]\frac{1}{(1-V_sx+x^2)^{r+1}}-\left[x^{k-r-2}\right]\frac{1}{(1-V_sx+x^2)^{r+1}}\right).$$

The proof of the theorem is completed.

We rewrite the main results of this paper as follows

$$W_n^{2k}(a,b)+W_{n+s}^{2k}(a,b)=\sum_{r=0}^{k}\Omega^r U_s^{2r}\left(\left[x^{k-r}\right]\frac{1}{(1-V_sx+x^2)^{r+1}}\right.$$
$$\left.-\left[x^{k-r-2}\right]\frac{1}{(1-V_sx+x^2)^{r+1}}\right)W_n^{k-r}(a,b)W_{n+s}^{k-r}(a,b). \qquad (8)$$

Corollary 2.3:

$$U_n^{2k}+U_{n+1}^{2k}=\sum_{r=0}^{k}\frac{\mathcal{D}^r V_k}{r!}U_n^{k-r}U_{n+1}^{k-r}.$$

Proof: Take $a=1, b=0, s=1$ in Theorem 2.2.

Corollary 2.4:

$$V_n^{2k}+V_{n+1}^{2k}=\sum_{r=0}^{k}(-1)^r(p^2-4)^r\frac{\mathcal{D}^r V_k}{r!}V_n^{k-r}V_{n+1}^{k-r}.$$

Proof: Take $a=0, b=1, s=1$ in Theorem 2.2.

We denote by $\sigma_i(n,k)$ the summation of all products of choosing i elements from $n+k-i+1, n+k-i+2, \ldots, n+2k-1$ but not containing any two consecutive elements, i.e.

$$\sigma_i(n,k)=\sum\prod_{t=1}^{i}(n+k-i+j_t)$$

where the summation is taken over all i-tuples with positive integer coordinates $(j_1, j_2, \ldots, j_i)$ such that $1\le j_1<j_2<\cdots<j_i\le k+i-1$ and $|j_r-j_s|\ge 2$ for $1\le r\ne s\le i$. For more details see [1].

Lemma 2.5: (*Feng and Zhang [1]*)

$$G_{sn}^{(k+1)}=\frac{1}{k!U_s(V_s^2-4)^k}\sum_{i=0}^{k}(-1)^i 2^i V_s^{k-i}\langle n\rangle_{k-i}\sigma_i(n,k)U_{s(n+k-i)}$$

where $\langle n\rangle_i=n(n+1)\ldots(n+i-1)$ *and* $\sum_{n\ge 0}G_{sn}^{(k)}x^{n-1}=\left(\frac{1}{1-V_sx+x^2}\right)^k$.

Proof: See [1].

We obtain the explicit expression of the coefficients $A_r(a,b;k,s)$ in Theorem 2.2 as follows.

Theorem 2.6:

$$A_r(a,b;k,s) = \frac{\Omega^r U_s^{r-1}}{r!(V_s^2-4)^r} \sum_{i=0}^{r} (-1)^i 2^i V_s^{k-i} \left[\langle k-r+1 \rangle_{r-i} \sigma_i(k-r+1,r) U_{s(k+1-r)} \right.$$

$$\left. - \langle k-r-1 \rangle_{r-i} \sigma_i(k-r-1,r) U_{s(k-1-r)} \right].$$

Proof: Combining Theorem 2.2 and Lemma 2.5, Theorem 2.6 follows.

ACKNOWLEDGMENT

The authors would like to thank an anonymous referee for many valuable suggestions that have improved this manuscript. This work is supported in part by the National Natural Science Foundation of China.

REFERENCES

[1] H. Feng and Z.Z. Zhang. "Computational Formulas for Convoluted Generalized Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **41.2** (2003): 144-151.

[2] P. J. Grabner and H. Prodinger. "Some Identities for the Chebyshev Polynomials." *Portugaliae Mathematica* **59** (2002): 311-314.

[3] R. S. Melham. "On Sums of Powers of Terms in a Linear Recurrence." *Portugaliae Mathematica* **56.4** (1999): 501-508.

AMS Classification Numbers: 05A15, 11B37, 11B39

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