A GENERALIZED SUMMATION RULE RELATED TO STIRLING NUMBERS

Xiqiang Zhao

Department of Mathematics, Ocean University of China, Qingdao Shandong 266071, P.R. China

Shuangshuang Ding

Department of Mathematics, Ocean University of China, Qingdao Shandong 266071, P.R. China

(Submitted July 2001–Final Revision October 2002)

1. THE GENERALIZED STIRLING NUMBERS

In this section and in section 2 and 3 we will generalize the summation rule obtained by L.C. Hsu [7], involving Stirling numbers of the second kind.

Given four real numbers a, b, α and β with $\alpha \neq 0$ and $\beta \neq 0$, L.C. Hsu and H.Q. Yu [11], L.C. Hsu and P.J. Shiue [9] defined the symmetric Stirling-type pairs ($\langle a, b, \alpha, \beta\rangle$- pairs, for short)

$$\{s^1, s^2\} = \{s^1(n,k), s^2(n,k)\} = \{s(n,k,a,b;\alpha,\beta), s(n,k,b,a;\beta,\alpha)\}$$

as follows:

$$(1+\alpha t)^{\frac{a-b}{\alpha}} \left(\frac{(1+\alpha t)^{\frac{\beta}{\alpha}} - 1}{\beta} \right)^k = k! \sum_{n \geq k} s^1(n,k) \frac{t^n}{n!}, \tag{1}$$

$$(1+\beta t)^{\frac{b-a}{\beta}} \left(\frac{(1+\beta t)^{\frac{\alpha}{\beta}} - 1}{\alpha} \right)^k = k! \sum_{n \geq k} s^2(n,k) \frac{t^n}{n!}, \tag{2}$$

where $\{s^1, s^2\}$ satisfies the following relations

$$((t+a)|\alpha)_n = \sum_{k=0}^{n} s^1(n,k)((t+b)|\beta)_k, \tag{3}$$

$$((t+a)|\beta)_n = \sum_{k=0}^{n} s^2(n,k)((t+a)|\alpha)_k, \tag{4}$$

where $(z|\alpha)_n = z(z-\alpha)\cdots(z-n\alpha+\alpha)$. We can obtain various $\langle a, b, \alpha, \beta\rangle$-pairs if a, b, α and β are various real numbers. We now list some of them below.

Let $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ denote the ordinary Stirling numbers of both kinds, respectively. Then $(-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ form the $\langle 0, 0; 1, 0\rangle$-pair and satisfy the following relations

$$(t)_n = \sum_{k=0}^{n} (-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix} t^k, \tag{5}$$

$$t^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (t)_k. \tag{6}$$

The degenerate Stirling numbers $s_1(n,k|\theta)$ and $s(n,k|\theta)$ in [1] form the $\langle 0,0;1,\theta\rangle$-pair and satisfy

$$(t)_n = \sum_{k=0}^{n}(-1)^{n+k}s_1(n,k|\theta)(t|\theta)_k, \tag{7}$$

$$(t|\theta)_n = \sum_{k=0}^{n} s(n,k|\theta)(t)_k. \tag{8}$$

The weighted Stirling numbers $R_1(n,k,\lambda)$ and $R_2(n,k,\lambda)$ in [2,3] form the $\langle -\lambda,0;1,0\rangle$-pair and satisfy

$$(t-\lambda)_n = \sum_{k=0}^{n}(-1)^{n+k}R_1(n,k,\lambda)t^k, \tag{9}$$

$$t^n = \sum_{k=0}^{n} R_2(n,k,\lambda)(t-\lambda)_k. \tag{10}$$

The non-central Stirling numbers $S_a(n,k)$ and $s_a(n,k)$ in [10] form the $\langle 0,-a;1,0\rangle$-pair and satisfy

$$(t)_n = \sum_{k=0}^{n} s_a(n,k)(t-a)^k, \tag{11}$$

$$(t-a)^n = \sum_{k=0}^{n} S_a(n,k)(t)_k. \tag{12}$$

The weighted degenerate Stirling numbers $s_1(n,k,\lambda|\theta)$ and $s(n,k,\lambda|\theta)$ in [6] form the $\langle 0,\lambda;1,\theta\rangle$-pair and satisfy

$$(t)_n = \sum_{k=0}^{n}(-1)^{n+k}s_1(n,k,(\lambda+\theta)|\theta)((t+\lambda)|\theta)_k, \tag{13}$$

$$((t+\lambda)|\theta)_n = \sum_{k=0}^{n} s(n,k,\lambda|\theta)(t)_k. \tag{14}$$

In addition, there are many generalized Stirling numbers that are not unified by (1) and (2), for example:

For any fixed real number a consider the Dickson polynomial $D_n(t,a)$ of degree $n \geq 1$ defined by

$$D_n(t,a) = \sum_{i=0}^{[n/2]} \frac{n}{n-i}\binom{n-i}{i}(-a)^i t^{n-2i},$$

where $D_0(t,a) = 2$ for all real t and a.

In [8], the Dickson-Stirling numbers of the first and second kinds, $s(n,k;a)$ and $S(n,k;a)$ respectively were defined by

$$(t-a)_n = \sum_{k=0}^{n} s(n,k;a)(D_k(t,a)-c_k), \tag{15}$$

$$D_n(t,a) - c_n = \sum_{k=0}^{n} S(n,k;a)(t-a)_k, \tag{16}$$

where $c_0 = 1$ and $c_k = 0$ for $k \geq 1$.

Here we cannot list Stirling numbers of all kinds. We now discuss some identities related to the above Stirling numbers. Actually, the following rules can be used for other Stirling numbers that satisfy the inverse relations (3)-(16).

2. SOME IDENTITIES ASSOCIATED WITH THE STIRLING NUMBERS

In [14], the following identities were obtained from (5) and (6)

$$\sum_{j=i}^{n} \left\{ \begin{matrix} n-i \\ j-i \end{matrix} \right\} (t)_j = \sum_{j=i}^{n} \left\{ \begin{matrix} n-i \\ j-i \end{matrix} \right\} \sum_{k=0}^{j} (-1)^{j+k} \begin{bmatrix} j \\ k \end{bmatrix} t^k = (t)_i (t-i)^{n-i}, \tag{17}$$

$$\sum_{j=i}^{n} (-1)^{n+j} \begin{bmatrix} n-i \\ j-i \end{bmatrix} t^j = \sum_{j=i}^{n} (-1)^{n+j} \begin{bmatrix} n-i \\ j-i \end{bmatrix} \sum_{k=0}^{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (t)_k = t^i (t)_{n-i}. \tag{18}$$

Similarly, from (3) and (4), we have

$$\begin{aligned} ((t+a)|\alpha)_i ((t+b-i\alpha)|\beta)_{n-i} &= \sum_{j=i}^{n} s^2(n-i, j-i)((t+a)|\alpha)_j \\ &= \sum_{j=i}^{n} s^2(n-i, j-i) \sum_{k=0}^{j} s^1(j,k)((t+b)|\beta)_k, \end{aligned} \tag{19}$$

and

$$\begin{aligned} ((t+b)|\beta)_i ((t+a-i\beta)|\alpha)_{n-i} &= \sum_{j=i}^{n} s^1(n-i, j-i)((t+b)|\beta)_j \\ &= \sum_{j=i}^{n} s^1(n-i, j-i) \sum_{k=0}^{j} s^2(j,k)((t+a)|\alpha)_k. \end{aligned} \tag{20}$$

Clearly, (17) and (18) are special cases of (20) and (19) with $a = b = \alpha = 0, \beta = 1$, respectively. By (7)-(16), we have the following identities in a similar way.

$$\begin{aligned} (t)_i ((t-i)|\theta)_{n-i} &= \sum_{j=i}^{n} s(n-i, j-i|\theta)(t)_j \\ &= \sum_{j=i}^{n} s(n-i, j-i|\theta) \sum_{k=0}^{j} (-1)^{j-k} s_1(j,k|\theta)(t|\theta)_k, \end{aligned} \tag{21}$$

$$\begin{aligned}(t|\theta)_i(t-i\theta)_{n-i} &= \sum_{j=i}^{n}(-1)^{n-j}s_1(n-i,j-i|\theta)(t|\theta)_j \\ &= \sum_{j=i}^{n}(-1)^{n-j}s_1(n-i,j-i|\theta)\sum_{k=0}^{j}s(j,k|\theta)(t)_k,\end{aligned} \tag{22}$$

$$\begin{aligned}(t-\lambda)_i(t-i)^{n-i} &= \sum_{j=i}^{n}R_2(n-i,j-i,\lambda)(t-\lambda)_j \\ &= \sum_{j=i}^{n}R_2(n-i,j-i,\lambda)\sum_{k=0}^{j}(-1)^{j-k}R_1(j,k,\lambda)t^k,\end{aligned} \tag{23}$$

$$\begin{aligned}t^i(t-\lambda)_{n-i} &= \sum_{j=i}^{n}(-1)^{n-j}R_1(n-i,j-i,\lambda)t^j \\ &= \sum_{j=i}^{n}(-1)^{n-j}R_1(n-i,j-i,\lambda)\sum_{k=0}^{j}R_2(j,k,\lambda)(t-\lambda)_k,\end{aligned} \tag{24}$$

$$\begin{aligned}(t)_i(t-a-i)^{n-i} &= \sum_{j=i}^{n}S_a(n-i,j-i)(t)_j \\ &= \sum_{j=i}^{n}S_a(n-i,j-i)\sum_{k=0}^{j}s_a(j,k)(t-a)^k,\end{aligned} \tag{25}$$

$$\begin{aligned}(t-a)^i(t)_{n-i} &= \sum_{j=i}^{n}s_a(n-i,j-i)(t-a)^j \\ &= \sum_{j=i}^{n}s_a(n-i,j-i)\sum_{k=0}^{j}S_a(j,k)(t)_k,\end{aligned} \tag{26}$$

$$\begin{aligned}&(t)_i(t+\lambda-i)_{n-i} \\ &= \sum_{j=i}^{n}s(n-i,j-i,\lambda|\theta)(t)_j \\ &= \sum_{j=i}^{n}s(n-i,j-i,\lambda|\theta)\sum_{k=0}^{j}(-1)^{j-k}s_1(j,k,(\lambda+\theta)|\theta)((t+\lambda)|\theta)_k,\end{aligned} \tag{27}$$

$$
\begin{aligned}
&((t+\lambda)|\theta)_i(t-i\theta)_{n-i} \\
&=\sum_{j=i}^{n}(-1)^{n-j}s_1(n-i,j-i,(\lambda+\theta)|\theta)((t+\lambda)|\theta)_j \\
&=\sum_{j=i}^{n}(-1)^{n-j}s_1(n-i,j-i,(\lambda+\theta)|\theta)\sum_{k=0}^{j}s(j,k,\lambda|\theta)(t)_k,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&(t-a)_i(D_{n-i}(t-i,a)-c_{n-i}) \\
&=\sum_{j=i}^{n}S(n-i,j-i;a)(t-a)_j \\
&=\sum_{j=i}^{n}S(n-i,j-i;a)\sum_{k=0}^{j}s(j,k;a)(D_k(t,a)-c_k).
\end{aligned}
\tag{29}
$$

3. THE GENERALIZED SUMMATION RULES

Theorem 1: Let $F(n,k)$ be a bivariate function defined for integers $n,k\geq 0$. If

$$\sum_{k=j}^{n}F(n,k)\binom{k}{j}=\varphi(n,j)\quad (j\geq 0),$$

then for given $m\geq i\geq 0$ we have the summation formula or the combinatorial identity

$$\sum_{k=0}^{n}F(n,k)(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!\left\{\begin{matrix}m-i\\ j-i\end{matrix}\right\}\varphi(n,j). \tag{30}$$

Proof:

$$
\begin{aligned}
\sum_{k=0}^{n}F(n,k)(k)_i(k-i)^{m-i}&=\sum_{k=0}^{n}F(n,k)\sum_{j=i}^{m}\left\{\begin{matrix}m-i\\ j-i\end{matrix}\right\}(k)_j \\
&=\sum_{j=i}^{m}\left\{\begin{matrix}m-i\\ j-i\end{matrix}\right\}\sum_{k=0}^{n}F(n,k)(k)_j \\
&=\sum_{j=i}^{m}\left\{\begin{matrix}m-i\\ j-i\end{matrix}\right\}\sum_{k=j}^{n}F(n,k)\binom{k}{j}j! \\
&=\sum_{j=i}^{m}\left\{\begin{matrix}m-i\\ j-i\end{matrix}\right\}\varphi(n,j)j!. \quad \square
\end{aligned}
$$

We now generalized (4)-(16) in [7] by (30).

$$\sum_{k=0}^{n}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n+1}{j+1}, \tag{31}$$

$$\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}p^jj!\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n}{j},\quad p+q=1,p>0, \tag{32}$$

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!2^{n-2j-1}\frac{n}{n-j}\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n-j}{j}, \tag{33}$$

$$\sum_{k=0}^{[n/2]}\binom{n+1}{2k+1}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!2^{n-2j}\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n-j}{j}, \tag{34}$$

$$\sum_{k=0}^{n}\binom{n-k}{s}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n+1}{s+j+1}, \tag{35}$$

$$\sum_{k=0}^{n}\binom{s+k}{s}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!\frac{n+1-j}{s+1+j}\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n+1}{j}\binom{n+1+s}{s}, \tag{36}$$

$$\sum_{k=0}^{n}(-4)^k\binom{n+k}{2k}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}(-1)^nj!2^{2j}\frac{2n+1}{2j+1}\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n+j}{2j}, \tag{37}$$

$$\begin{aligned}&\sum_{k=0}^{n}(-4)^k\frac{n}{n+k}\binom{n+k}{2k}(k)_i(k-i)^{m-i}\\&=\sum_{j=i}^{m}(-1)^nj!2^{2j}\frac{n}{n+j}\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n+j}{2j},\end{aligned} \tag{38}$$

$$\sum_{k=0}^{[n/2]}(-1)^k2^{n-2k}\binom{n-k}{k}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}(-1)^jj!\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n+1}{2j+1}, \tag{39}$$

$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}j!\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j}, \tag{40}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}(k)_i(k-i)^{m-i}=\sum_{j=i}^{m}(-1)^jj!\left\{\begin{matrix}m-i\\j-i\end{matrix}\right\}\binom{n}{j}^2, \tag{41}$$

$$\sum_{k=0}^{[n/2]} 2^{n-2k}\binom{n}{2k}\binom{2k}{k}(k)_i(k-i)^{m-i} = \sum_{j=i}^{m} j!\left\{\begin{matrix} m-i \\ j-i \end{matrix}\right\}\binom{2n-2j}{n}\binom{n}{j}, \tag{42}$$

$$\sum_{k=0}^{n}(k)_i(k-i)^{m-i}H_k = \sum_{j=i}^{m} j!\left\{\begin{matrix} m-i \\ j-i \end{matrix}\right\}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right), \tag{43}$$

where $H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}, k \geq 1$.

Theorem 2: Let $\binom{z}{n}_\alpha = \frac{(z|\alpha)_n}{n!}$ and $F(n,k)$ be a bivariate function defined for integers $n, k \geq 0$. If

$$\sum_{k=0}^{n} F(n,k)\binom{k+a}{j}_\alpha = G(n,j,a,\alpha),$$

then for given $m \geq i \geq 0$ we have

$$\sum_{k=0}^{n} F(n,k)\binom{k+a}{i}_\alpha\binom{k+b-i\alpha}{m-i}_\beta = \frac{1}{i!(m-i)!}\sum_{j=i}^{m} j!s^2(m-i,j-i)G(n,j,a,\alpha). \tag{44}$$

Proof:

$$\sum_{k=0}^{n} F(n,k)\binom{k+a}{i}_\alpha\binom{k+b-i\alpha}{m-i}_\beta$$

$$=\frac{1}{i!(m-i)!}\sum_{k=0}^{n} F(n,k)\sum_{j=i}^{m} s^2(m-i,j-i)((k+a)|\alpha)_j$$

$$=\frac{1}{i!(m-i)!}\sum_{j=i}^{m} s^2(m-i,j-i)\sum_{k=0}^{n} F(n,k)j!\binom{k+a}{j}_\alpha$$

$$=\frac{1}{i!(m-i)!}\sum_{j=i}^{m} j!s^2(m-i,j-i)G(n,j,a,\alpha). \quad \square$$

Let $a = b = \beta = 0$ and $\alpha = 1$. Then formula (44) becomes (30) in Theorem 1. Let $a = b = 0$ and $\alpha = 1$. It is readily seen that each of the formulas from (31) through (43) may be generalized to the form in which $(k)_i$ is replaced by $\binom{k}{i}$, $(k-i)^{m-i}$ by $\binom{k-i}{m-i}_\beta$ and $\left\{\begin{matrix} m-i \\ j-i \end{matrix}\right\}$ by $\frac{1}{i!(m-i)!}s^2(m-i,j-i)$. For example, (31) may be replaced by

$$\sum_{k=0}^{n}\binom{k}{i}\binom{k-i}{m-i}_\beta = \frac{1}{i!(m-i)!}\sum_{j=i}^{m} j!s^2(m-i,j-i)\binom{n+1}{j+1}. \tag{45}$$

In particular, for $\beta = 0$ and θ, (45) implies (31) and the following identity

$$\sum_{k=0}^{n} \binom{k}{i} \binom{k-i}{m-i}_{\theta} = \frac{1}{i!(m-i)!} \sum_{j=i}^{m} j! s(m-i, j-i|\theta) \binom{n+1}{j+1},$$

where $s(n, k|\theta)$ is the degenerate Stirling numbers in [1].

REFERENCES

[1] L. Carlitz. "Degenerate Stirling, Bernoulli and Eulerian Numbers." *Utilitas Math.* **15** (1979): 51-88.

[2] L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind I." *The Fibonacci Quarterly* **18** (1980): 145-162.

[3] L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind II." *The Fibonacci Quarterly* **18** (1980): 242-257.

[4] L. Comtet. *Advanced Combinatorics.* Reidel, Dordrecht, NL, 1974.

[5] R.B. Corcino. "Some Theorem on Generalized Stirling Numbers." *ARS Combinatoria* **60** (2000): 273-286.

[6] F.T. Howard. "Degenerate Weighted Stirling Numbers." *Discrete Mathematics* **57** (1985): 45-58.

[7] L.C. Hsu. "A Summation Rule Using Stirling Numbers of the Second Kind." *The Fibonacci Quarterly* **31** (1993): 256-262.

[8] L.C. Hsu, G.L. Mullen, P.J. Shiue. "Dickson-Stirling Numbers." *Proceeding of the Edinburgh Mathematical Society* **40** (1997): 409-423.

[9] L.C. Hsu and J.S. Shiue. "A Unified Approach to Generalized Stirling Numbers." *Advances in Applied Mathematics* **20** (1998): 366-384.

[10] M. Koutras. "Non-Central Stirling Numbers and Some Applications." *Discrete Mathematics* **42** (1982): 73-89.

[11] Xu Lizhi (L.C. Hsu) and Yu Hongquan. "A Unified Approach to a Class of Stirling-Type Pairs." *Appl. Math. A Journal of Chinese Universities* **12(B)** (1997): 225-232.

[12] D.E. Loeb. "A Generalization of the Stirling Numbers." *Discrete Mathematics* **103** (1992): 259-269.

[13] J. Riordan. *Combintorial Identities.* Wiley, New York, 1968.

[14] E.G. Whitehead. "Stirling Number Identities from Chromatic Polynomials." *Journal of Combinatorial Theorey, Series A* **24.3** May 1978.

AMS Classification Numbers: 05A19, 05A10

✠ ✠ ✠