ZEROS OF A CLASS OF FIBONACCI-TYPE POLYNOMIALS

Yi Wang
Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China

Mingfeng He
Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
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1. INTRODUCTION

Let a, b be two integers and $a \neq 0$. Consider a class of Fibonacci-type polynomials $G_n(x) = G_n(a, b; x)$ defined by the recursive relation

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x) \tag{1}$$

with the initial values $G_0(x) = a$ and $G_1(x) = x + b$. The polynomials $G_{n-1}(1, 0; x)$ and $G_n(2, 0; x)(n \geq 1)$ are just the usual Fibonacci polynomials $F_n(x)$ and the Lucas polynomials $L_n(x)$, respectively. Our concern in this paper is to study some of the properties of the zeros of the Fibonacci-type polynomials $G_n(a, b; x)$.

The zeros of $F_n(x)$ and $L_n(x)$ have been given explicitly by V. E. Hoggatt, Jr. and M. Bicknell [3] and N. Georgieva [2] (see MR52#5634 for corrections by Reviewer). However there are no general formulae for the zeros of the Fibonacci-type polynomials. There have been quite a few papers concerned the properties of the zeros of the Fibonacci-type polynomials in recent years. For example, G. A. Moore [8] and H. Prodinger [9] investigated the asymptotic behavior of the maximal real zeros of $G_n(-1, -1; x)$ respectively. The authors et al. [11] and F. Mátyás [5] investigated the same problem for $G_n(a, a; x)(a < 0)$ and $G_n(a, \pm a; x)(a \neq 0)$ respectively. In [6], F. Mátyás showed that the absolute values of complex zeros of polynomials $G_n(a, b; x)$ do not exceed $\max\{2, |a| + |b|\}$, which generalizes the result of P. E. Ricci [10] who investigated the problem in the case $a = b = 1$.

In the present paper, we first give a new bound $1 + \max\{|a|, |b|\}$ for the absolute values of the zeros of $G_n(a, b; x)$ by using the Gerŝchgorin's Cycle Theorem in §2. Our method can also obtain Mátyás' bound. Then in §3 we present a necessary and sufficient condition that $G_n(a, b; x)$ has real zeros. Finally in §4 we investigate the asymptotic behavior of the maximal real zeros of $G_n(a, b; x)$.

For our purposes, we need the Binet-form expression of $G_n(x)$. Following standard procedures, we easily obtain

$$G_n(x) = [c_1(x)\lambda_1^n(x) + c_2(x)\lambda_2^n(x)]/2\sqrt{x^2 + 4}, \tag{2}$$

where

$$\begin{cases} c_1(x) = a\sqrt{x^2 + 4} + (2 - a)x + 2b \\ c_2(x) = a\sqrt{x^2 + 4} - (2 - a)x - 2b \end{cases}$$

and

$$\begin{cases} \lambda_1(x) = (x + \sqrt{x^2 + 4})/2 \\ \lambda_2(x) = (x - \sqrt{x^2 + 4})/2 \end{cases}$$

are the roots of the associated characteristic equation of the sequence $G_n(x)$:

$$\lambda^2 - x\lambda - 1 = 0.$$

2. BOUNDS FOR ZEROS

Using the recursive relation (1) and an induction argument, it can be checked easily that

$$G_n(x) = \begin{vmatrix} x+b & -1 & & & & & \\ a & x & -1 & & & & \\ & 1 & x & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & x & -1 & \\ & & & & 1 & x & -1 \\ & & & & & 1 & x \end{vmatrix}$$

for $n \geq 2$. Thus $G_n(x)$ can be viewed as the characteristic polynomial of the $n \times n$ matrix

$$M_n = \begin{pmatrix} -b & 1 & & & & & \\ -a & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 0 & 1 & \\ & & & & -1 & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix},$$

and the zeros of $G_n(x)$ are therefore the eigenvalues of M_n. For the location of the eigenvalues of a matrix, we have the following well-known proposition due to Gerŝchgorin (see, e.g., [1], Theorem 9.1, p. 500). For a matrix $A = (a_{ij})$ of order n, define

$$r_i = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |a_{ij}|, \quad i = 1, 2, \ldots, n$$

and let Z_i denote the circle in the complex plane $\mathcal{C}$ with center a_{ii} and radius r_i (which is called the i^{th} Gerŝchgorin circle of A), that is

$$Z_i = \{z \in \mathcal{C} : |z - a_{ii}| \leq r_i\}.$$

Gerŝchgorin's Cycle Theorem: Let $A = (a_{ij})$ be a matrix of order n and let λ be an eigenvalue of A. Then λ belongs to one of the circles Z_i.

¿From this theorem it follows that the eigenvalues of the matrix M_n must be contained in the circles

$$|\lambda + b| \leq 1, \quad |\lambda| \leq |a| + 1, \quad |\lambda| \leq 2, \quad |\lambda| \leq 1.$$

Note that a is a nonzero integer. Hence we have the following.

Theorem 2.1: The zeros of $G_n(a,b;x)$ satisfy $|x| \le 1 + \max\{|a|, |b|\}$.

Note that the transpose M^T of a matrix M has the same eigenvalues as M. Hence the zeros of $G_n(a,b;x)$ are also the eigenvalues of the matrix M_n^T. Thus we obtain the following result by using Gerŝchgorin's Cycle Theorem again, which corresponds just with the result of Mátyás[6].

Theorem 2.2: The zeros of $G_n(a,b;x)$ satisfy $|x| \le \max\{2, |a| + |b|\}$.

Example 2.1: From Theorem 2.1 and 2.2 it follows that the zeros of the Fibonacci polynomials $F_n(x) = G_{n-1}(1,0;x)$ and the Lucas polynomials $L_n(x) = G_n(2,0;x)$ satisfy $|x| \le 2$, which can also be obtained directly from their expression given in [3].

3. EXISTENCE OF REAL ZEROS

In this section we investigate the existence of real zeros of $G_n(x)$. Denote by $R_n = R_n(a,b)$ the set of real zeros of $G_n(x) = G_n(a,b;x)$. In particular, $R_n = \emptyset$ if $G_n(x)$ has no real zeros. If $R_n \ne \emptyset$, then we denote by $r_n = r_n(a,b)$ the maximal real zero of $G_n(x)$. Note that the polynomial $G_n(x)$ is monic. If there exists a real number r such that $G_n(r) < 0$, then $R_n \ne \emptyset$ and $r_n > r$. Conversely, if $R_n \ne \emptyset$ and $r > r_n$, then $G_n(r) > 0$ (see, e.g., [8], Lemma 2.2A). When n is odd, we have $R_n \ne \emptyset$ since $G_n(x)$ is the polynomial of degree n. When n is even and $a \le 0$, we have $R_n \ne \emptyset$ for $n \ge 2$ since $G_n(0) = a$. What remains to consider is the case when n is even and $a > 0$.

The following lemma is a special case of Formula 4.1 in [8]

Lemma 3.1: If $R_n \ne \emptyset$, then $G_{n+2}(r_n) = -G_{n-2}(r_n)$.

Lemma 3.2: Suppose that $R_m \ne \emptyset$ and $G_{m+2}(r_m) < 0$ for some m. Then $R_{m+2k} \ne \emptyset$ and r_{m+2k} is monotonically increasing for $k = 0, 1, 2, \ldots$.

Proof: From $G_{m+2}(r_m) < 0$ it follows that $R_{m+2} \ne \emptyset$ and $r_{m+2} > r_m$. Now assume that $r_{m+2k} > r_{m+2(k-1)}$. Then $G_{m+2(k-1)}(r_{m+2k}) > 0$. So $G_{m+2(k+1)}(r_{m+2k}) < 0$ by Lemma 3.1. This yields $R_{m+2(k+1)} \ne \emptyset$ and $r_{m+2(k+1)} > r_{m+2k}$. Thus the statement holds by induction. □

Remark 3.1: Note that $r_1 = -b$ and $G_3(r_1) = -ab$. If $ab > 0$, then r_{2n-1} is monotonically increasing for $n \ge 1$.

Corollary 3.1: Suppose that $a > 0$ and that $R_{2m} \ne \emptyset$ for some m. Then $R_{2n} \ne \emptyset$ and r_{2n} is monotonically increasing for $n \ge m$.

Proof: Without loss of generality, we may assume that m is the smallest index such that $R_{2m} \ne \emptyset$. Then $m \ge 1$ and $G_{2(m-1)}(r_{2m}) > 0$. Thus $G_{2(m+1)}(r_{2m}) < 0$, and the statement follows from Lemma 3.2. □

Remark 3.2: If $a > 0$ and $b^2 - 4a \ge 0$, then $G_2(x) = x^2 + bx + a$ has real zeros. Thus $R_{2n} \ne \emptyset$ and r_{2n} is monotonically increasing for $n \ge 1$.

Denote

$$c(x) = c_1(x)c_2(x) = 4[(a-1)x^2 + (a-2)bx + a^2 - b^2]. \tag{3}$$

If $a = 1$ and $b \ne 0$, then $c(x) = 4(-bx + 1 - b^2)$ is linear and has unique zero:

$$\xi_0 = (1 - b^2)/b.$$

If $a \ne 1$, then $c(x)$ is quadratic with the discriminant

$$[(a-2)b]^2 - 4(a-1)(a^2 - b^2) = a^2(b^2 - 4a + 4).$$

Define

$$\Delta = b^2 - 4a + 4.$$

If $\Delta \geq 0$, then $c(x) = 0$ has two real roots:

$$\begin{cases} \xi_1 = [b(2-a) + a\sqrt{\Delta}]/2(a-1), \\ \xi_2 = [b(2-a) - a\sqrt{\Delta}]/2(a-1). \end{cases}$$

Theorem 3.1: Let $a > 0$. Then there exists n such that $R_{2n} \neq \emptyset$ if and only if

(i) $a = 1$ and $|b| > 1$, or

(ii) $a > 1$ and $\Delta > 0$.

Proof: Note that $G_n(a, b; x) = (-1)^n G_n(a, -b; -x)$. Hence $R_n(a, b) \neq \emptyset$ and $r \in R_n(a, b)$ imply that $R_n(a, -b) \neq \emptyset$ and $-r \in R_n(a, -b)$. Thus it suffices to consider the case $b \geq 0$. Now let $a \geq 1$ and $b \geq 0$. Then $G_{2n}(x) > 0$ for $x \geq 0$ since $G_{2n}(x)$ has nonnegative coefficients and positive constant term a. When $x < 0$, we have

$$c_1(x) = a\sqrt{x^2 + 4} + (2-a)x + 2b > (2-a)|x| + (2-a)x \geq 0.$$

If $c_2(x) \geq 0$ for all $x < 0$, then $G_{2n}(x) > 0$ for all $x < 0$ from (2). Hence $R_{2n} = \emptyset$. On the other hand, if $c_2(r) < 0$ for some $r < 0$, then $G_{2n}(r) < 0$ for sufficiently large n from (2) (since $|\lambda_1(r)| < 1$ and $|\lambda_2(r)| > 1$). Hence $R_{2n} \neq \emptyset$. Note that $c_2(x)$ has the same sign as that of $c(x) = c_1(x)c_2(x)$. So we need only check the sign of $c(x)$ for $x < 0$.

Case: $a = 1$. If $b = 0$ or $b = 1$, then $c(x) = 4$ or $c(x) = -4x$. Hence $R_{2n} = \emptyset$. If $b > 1$, then $c(x) < 0$ for $x \in (\xi_0, 0)$. Hence $R_{2n} \neq \emptyset$ for sufficiently large n.

Case: $a \geq 2$. If $\Delta \leq 0$, then $c(x) \geq 0$ from (3). Hence $R_{2n} = \emptyset$. If $\Delta > 0$, then $c(x)$ has two real zeros ξ_1 and ξ_2. Clearly, $c(x) < 0$ for $x \in (\xi_2, \min\{0, \xi_1\})$. Hence $R_{2n} \neq \emptyset$ for sufficiently large n.

Thus the proof of theorem is complete. □

Remark 3.3: If $a > 0$ and $b = 0$, then it follow easily from (1) by induction that

$$G_{2n}(x) = \sum_{i=0}^{n} a_i x^{2i}, \quad G_{2n+1}(x) = x\sum_{i=0}^{n} b_i x^{2i},$$

where a_i, b_i are nonnegative and $a_0 = b_0 = a > 0$. Thus $R_{2n} = \emptyset$ and $R_{2n+1} = \{0\}$. In particular, the Fibonacci polynomials $F_n(x) = G_{n-1}(1, 0; x)$ have no real zeros for n odd and have unique real zero 0 for n even, and the Lucas polynomials $L_n(x) = G_n(2, 0; x)$ have no real zeros for n even and have unique real zero 0 for n odd, which are the well-known results (see. e.g., [3]).

We conclude this section as follows.

Theorem 3.2: There exists m such that $R_n \neq \emptyset$ for all $n \geq m$ if and only if one of the following cases occurs:

(i) $a \leq 0$; (ii) $a = 1$ and $|b| > 1$; (iii) $a > 1$ and $b^2 > 4(a-1)$.

4. ASYMPTOTIC BEHAVIOR OF MAXIMAL REAL ZEROS

In [8], G. A. Moore considered the limiting behavior of the sequence $r_n(-1, -1)$ which are called "golden numbers". He confirmed an implication of computer analysis that $r_{2n-1}(-1, -1)$

is monotonically increasing and convergent to 3/2 from below, while $r_{2n}(-1,-1)$ is monotonically decreasing and convergent to 3/2 from above. In [11], the authors showed that for $a < 0$ the sequence $r_n(a,a)$ has the same monotonicity as $r_n(-1,-1)$ and instead of 3/2 the limit is $a(2-a)/(a-1)$. In this section we investigate the asymptotic behavior of $r_n(a,b)$ in general.

Lemma 4.1: Suppose that $\lim_{k\to+\infty} r_{n_k} = \xi$.

(a) If $\xi > 0$, then $c_1(\xi) = 0$.

(b) If $\xi < 0$, then $c_2(\xi) = 0$.

Proof: We only show (a) since (b) can be proved by a similar argument.

Assume that $\xi > 0$. Then there exists $r > 0$ such that $r_{n_k} > r$ for sufficiently large k. Thus

$$|\lambda_2(r_{n_k})| = \frac{1}{\lambda_1(r_{n_k})} = \frac{2}{r_{n_k} + \sqrt{r_{n_k}^2 + 4}} < \frac{2}{r + \sqrt{r^2+4}} < 1.$$

From (2) and $G_{n_k}(r_{n_k}) = 0$ it follows that

$$c_1(r_{n_k}) = (-1)^{n_k+1} c_2(r_{n_k}) \lambda_2^{2n_k}(r_{n_k}).$$

Letting $k \to +\infty$, we then obtain $c_1(\xi) = 0$, as required. □

Theorem 4.1: Suppose that $a < 0$.

(a) If $a + b \geq 0$, then r_{2n} is monotonically decreasing and convergent to 0 and r_{2n-1} is monotonically decreasing and convergent to ξ_2.

(b) If $a + b < 0$, then r_{2n} is monotonically decreasing and convergent to ξ_1 and r_{2n-1} is monotonically increasing and convergent to ξ_1.

Proof: For our purposes we need to make further exploration for $c_1(x)$ and $c_2(x)$. We have $c_1'(x) = 2 - a(1 - x/\sqrt{x^2+4}) > 0$. Also, $\lim_{x\to-\infty} c_1(x) = -\infty$ since $c_1(x) = a(\sqrt{x^2+4} + x) + 2(1-a)x + 2b$, and $\lim_{x\to+\infty} c_1(x) = +\infty$ since $c_1(x) = a(\sqrt{x^2+4} - x) + 2x + 2b$. Thus $c_1(x)$ is strictly increasing and has unique real zero ζ_1. Similarly, $c_2(x)$ is strictly decreasing and has unique real zero ζ_2. However, $c_1(x) + c_2(x) = 2a\sqrt{x^2+4} < 0$. In particular, $c_2(\zeta_1) < 0$. Hence $\zeta_1 > \zeta_2$. On the other hand, ζ_1 and ζ_2 are also the zeros of $c(x) = c_1(x)c_2(x)$. Consequently $\zeta_1 = \xi_1$ and $\zeta_2 = \xi_2$.

Even-Indices Sequence.

Note that $G_{2n}(0) = a < 0$ and $G_{2n}(\xi_1) = c_2(\xi_1)\lambda_1^{2n}(\xi_1) < 0$. Hence $r_{2n} > \max\{0, \xi_1\}$. It implies that

$$c_1(r_{2n}) > 0,\ c_2(r_{2n}) < 0,\ \lambda_1(r_{2n}) > 0,\ \lambda_2(r_{2n}) < 0.$$

By (2), $G_{2n-1}(r_{2n}) > 0$. Putting $x = r_{2n}$ in $G_{2n}(x) = xG_{2n-1}(x) + G_{2(n-1)}(x)$, we obtain $G_{2(n-1)}(r_{2n}) < 0$. This yields $r_{2(n-1)} > r_{2n}$. Hence the sequence r_{2n} is monotonically decreasing and therefore converging since the sequence is bounded by Theorem 2.1. Let $\lim_{n\to+\infty} r_{2n} = \xi$. Then $\xi \geq \max\{0, \xi_1\}$ since $r_{2n} > \max\{0, \xi_1\}$. Thus $\xi = \max\{0, \xi_1\}$ by Lemma 4.1(a).

Odd-Indices Sequence.

Assume that $x \geq \xi_1$ or $x \leq \xi_2$. Then $c_1(x)c_2(x) \leq 0$ by (3). Also, $c_1(x)$ and $c_2(x)$ are not equal to zero simultaneously since $c_1(x) + c_2(x) = 2a\sqrt{x^2+4} \neq 0$. On the other hand,

$\lambda_1(x)\lambda_2(x) = -1$. It follows that $G_{2n-1}(x) \neq 0$ from (2). Thus $r_{2n-1} \in (\xi_2, \xi_1)$. When $x \in (\xi_2, \xi_1)$, we have $c_1(x) < 0$ and $c_2(x) < 0$, so $G_{2n}(x) < 0$. Next we distinguish two cases.

Case 1: $\xi_1 \leq 0$. We have $r_{2n+1} < 0$. Putting $x = r_{2n+1}$ in $G_{2n+1}(x) = xG_{2n}(x)+G_{2n-1}(x)$, we obtain $G_{2n-1}(r_{2n+1}) < 0$.

Hence $r_{2n-1} > r_{2n+1}$ and the sequence r_{2n-1} is therefore monotonically decreasing. Thus r_{2n-1} converges to ξ_2 by Lemma 4.1(b).

Case 2: $\xi_1 > 0$. Let $r \in (\max\{0, \xi_2\}, \xi_1)$. Then $c_1(r) < 0, \lambda_1(r) > 1$ and $|\lambda_2(r)| < 1$. By (2), $G_{2n-1}(r) < 0$ for sufficiently large n. So $r_{2n-1} > r > 0$. Putting $x = r_{2n-1}$ in $G_{2n+1}(x) = xG_{2n}(x) + G_{2n-1}(x)$, we obtain $G_{2n+1}(r_{2n-1}) < 0$. It follows that r_{2n-1} is monotonically increasing from Lemma 3.2. Thus r_{2n-1} converges to ξ_1 by Lemma 4.1(a).

Finally, note that $c_1(0) = 2(a+b)$. Hence $\xi_1 \leq 0$ if $a+b \geq 0$ and $\xi_1 > 0$ if $a+b < 0$. This completes our proof. □

For the case $a > 0$, we give the following result but omit its proof for the sake of brevity.

Theorem 4.2: Suppose that $a = 1$ and $|b| > 1$.

(a) If $b < -1$, then r_{2n} is monotonically increasing and convergent to ξ_0 and r_{2n-1} is monotonically decreasing and convergent to ξ_0.

(b) If $b > 1$, then r_{2n} is monotonically increasing and convergent to 0 and r_{2n-1} is monotonically increasing and convergent to ξ_0.

Suppose that $a > 1$ and $\Delta > 0$.

(c) If $b < 0$, then r_{2n} is monotonically increasing and convergent to ξ_1 and r_{2n-1} is monotonically decreasing and convergent to ξ_1.

(d) If $0 < b < a$, then r_{2n} is monotonically increasing and convergent to ξ_1 and r_{2n-1} is monotonically increasing and convergent to 0.

(e) If $b \geq a$, then r_{2n} is monotonically increasing and convergent to 0 and r_{2n-1} is monotonically increasing and convergent to ξ_2.

Remark 4.1: From Theorem 4.1 and 4.2 we can conclude that the sequence r_n is convergent (to ξ) if and only if one of the following cases occurs:

Case: $a < 0$ and $a + b < 0$. $\xi = \xi_1$.

Case: $a < 0$ and $b = -a$. $\xi = 0$.

Case: $a = 1$ and $b < -1$. $\xi = \xi_0$.

Case: $a > 1, b < 0$ and $\Delta > 0$. $\xi = \xi_1$.

Case: $a = b > 2$. $\xi = 0$.

Remark 4.2: Recall $R_n(a, b) \neq \emptyset$ and $r \in R_n(a, b)$ imply that $R_n(a, -b) \neq \emptyset$ and $-r \in R_n(a, -b)$. Let $\overline{r}_n(a, b)$ denote the minimal real zero of $G_n(a, b; x)$. Then $\overline{r}_n(a, b) = -r_n(a, -b)$. Thus we may actually know the asymptotic behavior of the minimal real zeros of $G_n(x)$. For example, $r_{2n-1}(-1, -1)$ is monotonically decreasing and convergent to $3/2$, so $\overline{r}_{2n-1}(-1, 1)$ is monotonically increasing and convergent to $-3/2$.

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