ON WEIGHTED FIBONACCI AND LUCAS SUMS

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1. INTRODUCTION

T. Koshy wrote a fascinating book [2] on Fibonacci and Lucas numbers. The following summation formulas are well known. Let $S(m) := \sum_{j=1}^{n} j^m F_j$ and $T(m) := \sum_{j=1}^{n} j^m L_j$.

$$
\begin{aligned}
S(0) &= F_{n+2} - 1\\
T(0) &= L_{n+2} - 3\\
S(1) &= (n+1)F_{n+2} - F_{n+4} + 2\\
T(1) &= (n+1)F_{n+2} - F_{n+4} + 4.
\end{aligned}
$$

However these are less known formulas [1,2],

$$
\begin{aligned}
S(2) &= (n+1)^2F_{n+2} - (2n+3)F_{n+4} + 2F_{n+6} - 8\\
T(2) &= (n+1)^2L_{n+2} - (2n+3)L_{n+4} + 2L_{n+6} - 18\\
S(3) &= (n+1)^3F_{n+2} - (3n^2+9n+7)F_{n+4} + (6n+12)F_{n+6} - 6F_{n+8} + 50\\
T(3) &= (n+1)^3L_{n+2} - (3n^2+9n+7)L_{n+4} + (6n+12)L_{n+6} - 6L_{n+8} + 112\\
S(4) &= (n+1)^4F_{n+2} - (4n^3+18n^2+28n+15)F_{n+4} + (12n^2+48n+50)F_{n+6}\\
&\quad - (24n+60)F_{n+8} + 24F_{n+10} - 416\\
T(4) &= (n+1)^4L_{n+2} - (4n^3+18n^2+28n+15)L_{n+4} + (12n^2+48n+50)L_{n+6}\\
&\quad - (24n+60)L_{n+8} + 24L_{n+10} - 930.
\end{aligned}
$$

He mentioned a few interesting properties from these formulas without proof. For example,
(1) Both $S(m)$ and $T(m)$ contain $m+2$ terms.
(2) The leading term in $S(m)$ is $(n+1)^mF_{n+2}$, and that in $T(m)$ is $(n+1)^mL_{n+2}$.
(3) The subscripts in the Fibonacci and Lucas sums increase by 2, while the exponents of n in each coefficient decrease by one.

The aim of this note is to establish summation formulas for $S(m)$ and $T(m)$ explicitly. We use the differential operator method, which is discussed in [1]. First we introduce the operator $\nabla f(x)$ which is defined by

$$\nabla f(x) = x\frac{df(x)}{dx}$$

$$\nabla^n f(x) = \nabla(\nabla^{n-1} f(x)), \nabla^0 f(x) = f(x).$$

The following lemma is well known, for example [3], and can be proved by straightforward induction.

Lemma 1: *If $f(x)$ is differentiable then*

$$\nabla^n f(x) = \sum_{j=1}^{n} S(n,j) x^j f^{(j)}(x) \tag{1}$$

where $S(n,j)$ is the Stirling numbers of the second kind; they are defined by

$$x^n = \sum_{j=0}^{n} S(n,j)(x)_j \tag{2}$$

$$\textit{with} \quad (x)_j = x(x-1)\dots(x-j+1), (x)_0 = 1.$$

Let $f(x) = \sum_{j=1}^{n} x^j$ and $g(x) = \frac{1-x^{n+1}}{1-x}$, where $x \neq 1$. We have

$$\nabla f(x) = \sum_{j=1}^{n} j x^j \text{ and } \nabla f(x) = \nabla g(x).$$

More generally, we have

$$\nabla^m f(x) = \nabla^m g(x) = \sum_{j=1}^{n} j^m x^j, \quad \text{for} \quad m \geq 1.$$

Using the Binet formula for Lucas numbers,

$$T(m) = \sum_{j=1}^{n} j^m L_j = \sum_{j=1}^{n} j^m (\alpha^j + \beta^j) = (\nabla^m g(x))_{x=\alpha} + (\nabla^m g(x))_{x=\beta}$$

$$= (\nabla^m (g_0(x) - g_{n+1}(x)))_{x=\alpha} + (\nabla^m (g_0(x) = g_{n+1}(x)))_{x=\beta}$$

where $g_t(x) = \frac{x^t}{1-x}$. Suppose we have a formula for $T(m)$. Since we can obtain F_i from L_i by changing β^i to $-\beta^i$ and then dividing the difference by $\sqrt{5}$, we can find a formula for $S(m)$ from $T(m)$. We consider $T(m)$ in detail.

2. CONSTANT TERM

The j^{th} derivative of $g_0(x)$ is expressed by

$$g_0^{(j)}(x) = \frac{j!}{(1-x)^{j+1}}.$$

By Lemma 1 we have

$$\nabla^m g_0(x) = \sum_{j=1}^{m} S(m,j) x^j g_0^{(j)}(x) = \sum_{j=1}^{m} \frac{j! S(m,j) x^j}{(1-x)^{j+1}}.$$

Since $\frac{1}{1-\alpha} = -\alpha$ and $\frac{1}{1-\beta} = -\beta$, we have the constant term of $T(m)$,

$$(\nabla^m g_0(x))_{x=\alpha} + (\nabla^m g_0(x))_{x=\beta} = \sum_{j=1}^{m} (-1)^{j+1} j! S(m,j) L_{2j+1}.$$

3. GENERAL TERM

Consider the general term for $T(m)$,

$$\nabla^m g_{n+1}(x) = \sum_{j=1}^{m} S(m,j) x^j g_{n+1}^{(j)}(x)$$

$$= \sum_{j=1}^{m} S(m,j) x^j \sum_{i=0}^{j} i! \binom{j}{i} \frac{(x^{n+1})^{(j-i)}}{(1-x)^{i+1}}$$

$$= \sum_{j=1}^{m} S(m,j) x^j \sum_{i=0}^{j} i!(n+1)_{j-1} \binom{j}{i} \frac{x^{n+i-j+1}}{(1-x)^{i+1}}.$$

So we have

$$(\nabla^m g_{n+1}(x))_{x=\alpha} + (\nabla^m g_{n+1}(x))_{x=\beta}$$

$$= \sum_{j=1}^{m} S(m,j) \left\{ \sum_{i=0}^{j-1} (-1)^{i+1} i!(n+1)_{j-i} \binom{j}{i} L_{n+2i+2} \right\} + \sum_{j=1}^{m} (-1)^{j+1} j! S(m,j) L_{n+2j+2}$$

$$= -S(m,1)(n+1)_1 \binom{1}{0} L_{n+2}$$

$$+ \left\{ -S(m,2)(n+1)_2 \binom{2}{0} L_{n+2} + 1!S(m,2)(n+1)_1 \binom{2}{1} L_{n+4} \right\}$$

$$+ \left\{ -S(m,3)(n+1)_3 \binom{3}{0} L_{n+2} + 1!S(m,3)(n+1)_2 \binom{3}{1} L_{n+4} \right.$$

$$\left. -2!S(m,3)(n+1)_1 \binom{3}{2} L_{n+6} \right\} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \left\{ -S(m,m)(n+1)_m \binom{m}{0} L_{n+2} + 1!S(m,m)(n+1)_{m-1} \binom{m}{1} L_{n+4} + \dots \right.$$

$$\left. \dots + (-1)^m (m-1)! S(m,m)(n+1)_1 \binom{m}{m-1} L_{n+2m} \right\}$$

$$+ 1!S(m,1)L_{n+4} - 2!S(m,2)L_{n+6} + 3!S(m,3)L_{n+8} + \dots$$
$$\dots + (-1)^m (m-1)! S(m,m-1) L_{n+2m} + (-1)^{m+1} m! S(m,m) L_{n+2m+2}.$$

Rearranging terms vertically we have

$$(\nabla^m g_{n+1}(x))_{x=\alpha} + (\nabla^m g_{n+1}(x))_{x=\beta}$$

$$= \left(-\sum_{j=1}^{m} S(m,j)(n+1)_j \right) L_{n+2}$$

$$+ \sum_{t=2}^{m} \left\{ \sum_{j=t}^{m} (-1)^t (t-1)! \binom{j}{t-1} (n+1)_{j-t+1} S(m,j) + (-1)^t (t-1)! S(m,t-1) \right\} L_{n+2t}$$

$$+ (-1)^{m+1} m! S(m,m) L_{n+2m+2}.$$

We immediately have

Theorem 1: *For* $m \geq 2$

$$\sum_{j=1}^{n} j^m L_j = (n+1)^m L_{n+2}$$

$$- \sum_{t=2}^{m} \left\{ \sum_{j=t}^{m} (-1)^t (t-1)! \binom{j}{t-1} (n+1)_{j-t+1} S(m,j) + (-1)^t (t-1)! S(m,t-1) \right\} L_{n+2t}$$

$$+ (-1)^m m! L_{n+2m+2} + \sum_{j=1}^{m} (-1)^{j+1} j! S(m,j) L_{2j+1}.$$

Replacing L_j with F_j, yields

Theorem 2: *For* $m \geq 2$

$$\sum_{j=1}^{n} j^m F_j = (n+1)^m F_{n+2}$$

$$- \sum_{t=2}^{m} \left\{ \sum_{j=t}^{m} (-1)^t (t-1)! \binom{j}{t-1} (n+1)_{j-t+1} S(m,j) + (-1)^t (t-1)! S(m,t-1) \right\} F_{n+2t}$$

$$+ (-1)^m m! F_{n+2m+2} + \sum_{j=1}^{m} (-1)^{j+1} j! S(m,j) F_{2j+1}.$$

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