CHEBYSHEV AND PELL CONNECTIONS

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1. BEGINNING

Polynomials $R_n^{(r,u)}(x)$ and $S_n^{(r,u)}(x)$:

Polynomials $R_n^{(r,u)}(x)$ were defined in [3] by the recurrence

$$R_n^{(r,u)}(x) = (x+2)R_{n-1}^{(r,u)}(x) - R_{n-2}^{(r,u)}(x) \quad (n \geq 2) \tag{1.1}$$

with

$$R_0^{(r,u)}(x) = u,\ R_1^{(r,u)}(x) = x + r + u. \tag{1.2}$$

Their properties were investigated in conjunction with coefficients $c_{n,k}^{(r,u)}$ [3, Table 1] where

$$R_n^{(r,u)}(x) = \sum_k^n c_{n,k}^{(r,u)} x^k \tag{1.3}$$

with some conditions attached to the $c_{n,k}^{(r,u)}$ [3, (2.4)-(2.10)].

Later, in [4], a similar examination was made of the polynomials $S_n^{(r,u)}(x)$ defined by

$$S_n^{(r,u)}(x) = (x+2)S_{n-1}^{(r,u)}(x) + S_{n-2}^{(r,u)}(x) \quad (n \geq 2) \tag{1.4}$$

with

$$S_0^{(r,u)}(x) = u,\ S_1^{(r,u)}(x) = x + r + u, \tag{1.5}$$

where

$$S_n^{(r,u)}(x) = \sum_K^n d_{n,k}^{(r,u)} x^k \tag{1.6}$$

with conditions imposed on the coefficients $d_{n,k}^{(r,u)}$ [4, (3.4)-(3.7) Table 2].

Pell Convolutions:

(i) In [4], the coefficients $d_{n,k}^{(r,u)}$ for $S_n^{(r,u)}(x)$ in (1.6) were related to Pell convolution numbers $P_n^{(m)}$ which are displayed in [4, Table 1].

(ii) Furthermore, in [6], the nine $S_n^{(r,u)}(x)$ were expressed as sums of Pell convolutions, the proofs involving $d_{n,k}^{(r,u)}$.

Chebyshev Convolutions:

Here, we establish theories corresponding to (i) and (ii) for $R_n^{(r,u)}(x)$, $c_{n,k}^{(r,u)}$, and the m^{th} *Chebyshev convolutions* $U_n^{(m)}$ for the Chebyshev polynomials $U_n(x)$ defined by [2], where $n \to n+1$,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \qquad (n \geq 2), U_0(x) = 0, U_1(x) = 1. \tag{1.7}$$

Associated with $U_n^{(m)}$ are the m^{th} *Chebyshev convolutions* $T_n^{(m)}$ for the Chebyshev polynomials $T_n(x)$ given in [2] by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (n \geq 2), T_0(x) = 2, T_1(x) = 2x. \tag{1.8}$$

Observe from (1.7), (1.8) that, when $x = 1, U_n(1) \equiv U_n, T_n(1) \equiv T_n$,

$$U_n - U_{n-1} = \frac{1}{2}T_n. \tag{1.9}$$

Clearly,

$$\{U_n\} = \{0, 1, 2, 3, 4, \cdots\}, \tag{1.10}$$

$$\{T_n\} = \{2, 2, 2, 2, 2, \cdots\}. \tag{1.11}$$

Just as the $R_n^{(r,u)}(x)$ are expressible as Morgan-Voyce polynomials [3], so the $S_n^{(r,u)}(x)$ may be expressed in terms of quasi Morgan-Voyce polynomials [4].

Coefficients $d_{n,k}^{(r,u)}$ and $c_{n,k}^{(r,u)}$:

One can spot, though not easily, the important connection, involving *interchange* of superscripts,

$$d_{n,k}^{(r,u)} = d_{n-1,k}^{(u,r)} + c_{n,k}^{(u,r)} + c_{n-1,k}^{(u,r)} \qquad (k > n-1). \tag{1.12}$$

The restriction $k > n-1$ in (1.12) is imposed by virtue of $c_{n,n}^{(r,u)} = d_{n,n}^{(r,u)} = 1, c_{n,n-1}^{(r,u)} = d_{n,n-1}^{(r,u)} = 2(n-1) + r + u$ from [3, Table 1] and [4, Table 2]. It precludes any simple link between $R_n^{(r,u)}(x)$ and $S_n^{(r,u)}(x)$ along the lines of (1.12).

Question: Can we find, if it exists, a formula involving binomial coefficients for $d_{n,k}^{(r,u)}$ analogous to that for $c_{n,k}^{(r,u)}$ in [2, (2.11)]?

Purpose of this Paper:

Basically, the aim of our endeavours is to discover sets of relationships, particularly for convolutions, amoung T_n, U_n, P_n, and Q_n (the Pell-Lucas numbers (6.11)). In the process, further data about $R_n^{(r,u)}(x)$ and $S_n^{(r,u)}(x)$ are determined.

2. CHEBYSHEV CONVOLUTIONS $U_n^{(m)}$

¿From (1.7) when $x = 1$, we define $U_n^{(m)} \equiv U_n^{(m)}(1)$ thus:

Definition:

$$\sum_{n=1}^{\infty} U_n^{(m)} y^{n-1} = [1 - (2y - y^2)]^{-(m+1)}, U_0^{(m)} = 0 \tag{2.1}$$

$$= (1-y)^{-2(m+1)}, \tag{2.1a}$$

with $U_n^{(0)} \equiv U_n$.

Calculation in (2.1) leads to the following display for $U_n^{(m)}$ (Table 1).

$n\backslash m$	0	1	2	3	4
1	1	1	1	1	1
2	2	4	6	8	10
3	3	10	21	36	55
4	4	20	56	120	220
5	5	35	126	330	715

. . . .

Table 1. Chebyshev Convolution Numbers $U_n^{(m)}$.

These convolution numbers $U_n^{(m)}$ may be verified using the table in [1] derived from Cauchy products for sequences.

Basic Properties of $U_n^{(m)}, c_{n,k}^{(r,u)}$:

Analogously to the corresponding results (2.1), (2.4), (2.5), (3.7), and Theorem 4 in [4] - but see also [6, (1.6)] - we derive

$$U_n^{(m)} = 2U_{n-1}^{(m)} - U_{n-2}^{(m)} + U_n^{(m-1)} \qquad (\textit{recurrence}) \tag{2.2}$$

$$U_{n+1-m}^{(m)} - U_{n-1-m}^{(m)} = \frac{n}{m} U_{n+1-m}^{(m-1)} \tag{2.3}$$

$$U_n^{(m)} - U_{n-1}^{(m)} = \frac{n+2m-1}{2m} U_n^{(m-1)} \tag{2.4}$$

$$c_{n,k}^{(r,u)} = 2c_{n-1,k}^{(r,u)} - c_{n-2,k}^{(r,u)} + c_{n-1,k-1}^{(r,u)} \qquad (n \geq 2, k \geq 1 \tag{2.5}$$

$$c_{n,k}^{(r,u)} = U_{n-k+1}^{(k-1)} + rU_{n-k}^{(k)} + u \cdot \frac{n-k}{2k} U_{n-k+1}^{(k-1)}. \tag{2.6}$$

Observe that (2.2), (2.3), and (2.5) differ from the corresponding expressions for $P_n^{(m)}$ in [4] by having a centrally located + sign replaced by a − sign, whereas (2.4) and (2.6) do not exhibit this difference. These variations reflect the differences in sign in the definitions of $R_n^{(r,u)}(x)$ and $S_n^{(r,u)}(x)$ in (1.1) and (1.4) respectively.

3. $R_n^{(r,u)}(x)$ SUMMATIONS IN TERMS OF $U_n^{(k)}$

These summation formulas (Theorems 1-9) are analogous to corresponding statements in [6]. Therefore, their proofs will not be offered. Connections with Morgan-Voyce polynomials $B_n(x), b_n(x), C_n(x), c_n(x)$ appear as appropriate [3, p. 234].

Theorem 1: $R_n^{(1,1)}(x) = \sum_{k=0}^{n} U_{n+1-k}^{(k)} x^k = B_{n+1}(x)$.

Theorem 2: $R_n^{(0,0)}(x) = \sum_{k=0}^{n-1} U_{n-k}^{(k)} x^{k+1}$.

Theorem 3: $R_n^{(0,1)}(x) = \sum_{k=1}^{n} \frac{n+k}{k} U_{n+1-k}^{(k-1)} x^k + \frac{1}{2} T_n = b_{n+1}(x)$.

Theorem 4: $R_n^{(1,0)}(x) = \sum_{k=1}^{n} \left(U_{n-k}^{(k)} + U_{n+1-k}^{(k-1)} \right) x^k + U_n$.

Theorem 5: $R_n^{(2,2)}(x) = \sum_{k=1}^{n} \left(2U_{n+1-k}^{(k)} - U_{n+1-k}^{(k-1)} \right) x^k + 2U_{n+1}$.

Theorem 6: $R_n^{(1,2)}(x) = \sum_{k=0}^{n} \left(U_{4-k}^{(k)} + U_{3-k}^{(k)} + U_{2-k}^{(k)} \right) x^k$.

Theorem 7: $R_n^{(0,2)}(x) = \sum_{k=1}^{n} \frac{n}{k} U_{n+1-k}^{(k-1)} x^k + T_n$.

Theorem 8: $R_n^{(2,0)}(x) = \sum_{k=1}^{n} \left(2U_{n-k}^{(k)} + U_{n+1-k}^{(k-1)} \right) x^k + 2U_n$.

Theorem 9: $R_n^{(2,1)}(x) = \sum_{k=1}^{n} \left(2U_{n-k}^{(k)} + \frac{n+k}{2k} U_{n+1-k}^{(k-1)} \right) x^k + 2U_{n+1} + \frac{1}{2} T_n$.

Recall (1.10), (1.11) that $U_n = n, T_n = 2$ appearing in the above enunciations.

Examples:

$$R_3^{(2,1)}(x) = x^3 + 7x^2 + 14x + 7 = c_4(x)$$

$$R_3^{(0,2)}(x) = x^3 + 6x^2 + 9x + 2 = C_3(x).$$

Remarks: When applying basic formulas for $d_{n,k}^{(r,u)}$ corresponding to those for $c_{n,k}^{(r,u)}$, bear in mind the substitution of a $-$ in (2.2) for $a+$ in [4, (2.1)] in the middle term on the right-hand side. This effects the occurrence of $-$ in (2.3) instead of a $+$ in [4, (2.4)]. Thus, Theorems 5, 6, and 7, whose proofs involve the use of (2.3), will necessarily contain a slight variation from the corresponding proofs in [6; Theorems 5, 6, 7].

As a theorem of representative difficulty from Theorems 1-9, we choose to prove Theorem 5 (noting the Question at the completion of the proof).

But firstly observe from [3, (2.7)] that, for $r = u - 2$,

$$c_{n,0}^{(2,2)} = 2n + 2 = 2U_{n+1} \tag{3.1}$$

by (1.10). In fact (cf. [4, Theorem 1]),

$$c_{n,0}^{(r,u)} = U_n r + \frac{1}{2} T_n u \left(\frac{1}{2} T_n = U_n - U_{n-1} \right).$$

Proof of Theorem 5: Consider $2R_n^{(1,1)}(x)$ for k, then subtract $R_{n-1}^{(1,1)}(x)$ for $k-1$.

Accordingly, from (2.3), using (2.6) and simplifying, we have

$$2c^{(1,1)}_{n,k} - c^{(1,1)}_{n-1,k-1} = 2U^{(k)}_{n-k} + \frac{n+k}{k}U^{(k-1)}_{n-k+1} - U^{(k-1)}_{n-k} - \frac{n-2+k}{2(k-1)}U^{(k-2)}_{n-k+1}$$

$$= 2U^{(k)}_{n-k} + \frac{n}{k}U^{(k-1)}_{n-k+1} \text{ by (2.4)} \quad \cdots\cdots\cdots\cdots (A)$$

$$= 2U^{(k)}_{n-k+1} - U^{(k-1)}_{n+1-k} \text{ also, by (2.4)} \quad \cdots\cdots (B)$$

$$= U^{(k-1)}_{n-k+1} + 2U^{(k)}_{n-k} + 2\frac{(n-k)}{2k}U^{(k-1)}_{n-k+1} \text{ from } (A)$$

$$= c^{(2,2)}_{n,k} \text{ by (2.6).}$$

Applying (B) we have the theorem, where $2U_{n+1}$ orginates with $k=0$. (Refer to (3.1) also).

Question: How do we know to obtain $c^{(2,2)}_{n,k} = 2c^{(1,1)}_{n,k} - c^{(1,1)}_{n-1,k-1}$?

Response: Short tables for $c^{(r,u)}_{n,k}$ need to be constructed using [3, Table 1]. In Theorem 5, we commence with $r=u=2$. A little detection with practice, enables us to "spot" the required relationship.

4. CHEBYSHEV CONVOLUTION POLYNOMIALS $U^{(m)}_n(x)$

Generating Function Definition:

Extending (2.1), we define the m^{th} *Chebyshev convolution polynomial* $U^{(m)}_n(x)$ of $U_n(x)$ by a generating function:

$$\sum_{n=1}^{\infty} U^{(m)}_n(x)y^{n-1} = [1-(2xy-y^2)]^{-(m+1)}, U^{(m)}_0(x) = 0. \tag{4.1}$$

When $m=0, U^{(0)}_n(x) \equiv U_n(x)$. Putting $x=1$ in (4.1) leads us back to (2.1).

Simplest expressions:

$\underline{m=0}$: $U^{(0)}_1(x) = 1, U^{(0)}_2(x) = 2x, U^{(0)}_3(x) = 4x^2-1, U^{(0)}_4(x) = 8x^3-4x,$

$$U^{(0)}_5(x) = 16x^4 - 12x^2 + 1, \cdots . \tag{4.2}$$

$\underline{m=1}$: $U^{(1)}_1(x) = 1, U^{(1)}_2(x) = 4x, U^{(1)}_3(x) = 12x^2-2, U^{(1)}_4(x) = 32x^3-12x,$

$$U^{(1)}_5(x) = 80x^4 - 48x^2 + 3, \cdots . \tag{4.3}$$

$\underline{m=2}$: $U^{(2)}_1(x) = 1, U^{(2)}_2(x) = 6x, U^{(2)}_3(x) = 24x^2-3, U^{(2)}_4(x) = 80x^3-24x,$

$$U^{(2)}_5(x) = 240x^4 - 120x^2 + 6, \cdots . \tag{4.4}$$

Generally,

$$U_1^{(m)}(x) = 1, U_2^{(m)}(x) = \binom{m+1}{1} 2x, U_3^{(m)}(x) = \binom{m+2}{2}(2x)^2 - \binom{m+1}{1},$$

$$U_4^{(m)}(x) = \binom{m+3}{3}(2x)^3 - 2\binom{m+2}{2} 2x, \tag{4.5}$$

$$U_5^{(m)}(x) = \binom{m+4}{4}(2x)^4 - 3\binom{m+3}{3}(2x)^2 + \binom{m+2}{2}, \cdots .$$

Combinatorial Definition:

$$U_n^{(m)}(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (-1)^j \binom{n+m-1-j}{m}\binom{n-1-j}{j}(2x)^{n-1-2j}. \tag{4.6}$$

The equivalence of (4.1) and (4.6) is now established.

Proof of (4.6) from (4.1): Expand (4.1) to obtain the general term corresponding to y^{n-1}. We have

$$\binom{m+k}{k}(2x-y)^k y^k = \sum_{j=0}(-1)^j \binom{m+k}{k}\binom{k}{j}(2x)^{k-j}y^{k+j}$$

$$= \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (-1)^j \binom{m+n-1-j}{m}\binom{n-1-j}{j}(2x)^{n-1-2j}y^{n-1} \cdots (I)$$

on writing $k+j=n-1$, i.e., $k=n-1-j$. Equating coefficients of y^{n-1} in (4.1) and (I) we are left with (4.6) as required.

Alternatively, to flush out (4.6), we extend the closed forms [8, (1.7), (4.9), (6.6)] for $P_n^{(m)}(x)$ where $m = 0, 1, 2$ respectively, and then adjust them to apply to $U_n^{(m)}(x)$.

Accordingly, the previously unrecorded closed form expression for $P_n^{(m)}(x)$ is revealed as

$$P_n^{(m)}(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n+m-1-j}{m}\binom{n-1-j}{j}(2x)^{n-1-2j}. \tag{4.7}$$

Cauchy Summation Definition for $U_n^{(m)}(x)$:

$$U_n^{(m)}(x) = \sum_{j=1}^{n} U_j(x) U_{n+1-j}^{(m-1)}(x). \tag{4.8}$$

Some of the simplest expressions in (4.2)-(4.4) may now be checked by means of (4.6) and (4.8).

5. CHEBYSHEV CONVOLUTION POLYNOMIALS $T_n^{(m)}(x)$

Designate the m^{th} *Chebyshev polynomial* of $T_n(x)$ by $T_n^{(m)}(x)$.

Generating Function Definition:

$$\sum_{n=1}^{\infty} T_n^{(m)}(x)y^{n-1} = (2x-2y)^{m+1}[1-(2xy-y^2)]^{-(m+1)}, T_0^{(m)}(x) = 0. \tag{5.1}$$

When $x = 1$, we have the m^{th} *Chebyshev convolution numbers* $T_n^{(m)}$, where $T_n^{(m)}(1) \equiv T_n^{(m)}$, and $T_n^{(0)} \equiv T_n$ given by

$$\sum_{n=1}^{\infty} T_n^{(m)} y^{n-1} = 2^{m+1}(1-y)^{-(m+1)} \tag{5.1a}$$

so that (1.11)

$$\left\{T_n^{(0)}\right\} = 2\{1,1,1,1,\dots\}. \tag{5.1b}$$

Cauchy Summation Definition:

$$T_n^{(m)}(x) = \sum_{j=1}^{n} T_j(x) T_{n+1-j}^{(m-1)}(x). \tag{5.2}$$

¿From (5.1) and (5.2), simple expressions of $T_n^{(m)}(x)$ for small values of m and n may be readily calculated. Clearly, by (5.2) with $x = 1$,

$$T_n^{(m)} = 2\sum_{j=1}^{n} T_{n+1-j}^{(m-1)}. \tag{5.2a}$$

Relationship between $U_n^{(m)}$ and $T_n^{(m)}$:

Coming to the intrinsic connection between $U_n^{(m)}$ and $T_n^{(m)}$, we learn that this is determined by the parity of the superscript.

Theorem 10:

(i) $T_n^{(2m+1)} = 2^{2(m+1)} U_n^{(m)}$,

(ii) $T_n^{(2m)} = 2^{2m+1} \sum_{j=1}^{n} U_j^{(m-1)}$.

Proof:

(i)

$$\sum_{n=1}^{\infty} T_n^{(2m+1)} y^{n-1} = 2^{2(m+1)}(1-y)^{-2(m+1)} \qquad \text{by (5.1a)}$$

$$= 2^{2(m+1)} \sum_{n=1}^{\infty} U_n^{(m)} y^{n-1} \qquad \text{by (2.1a)}$$

Equate coefficients of y^{n-1}. Then (i) ensues.

(ii)
$$\begin{aligned}\sum_{n=1}^{\infty} T_n^{(2m)} y^{n-1} &= 2^{2m+1}(1-y)^{-(2m+1)} && \text{by (5.1a)}\\ &= 2^{2m+1}(1-y)^{-1}\sum_{n=1}^{\infty} U_n^{(m-1)} y^{n-1} && \text{by (2.1a)}\\ &= 2^{2m+1}\sum_{n=1}^{\infty}\sum_{j=1}^{n} U_j^{(m-1)} y^{n-1}.\end{aligned}$$

Coefficients of y^{n-1} being equated, we derive (ii).

Examples:

(i) $T_n^{(3)} = 2^4 U_n^{(1)}$, i.e. $T_5^{(3)} = 16 \cdot 35 = 560$.

(ii) $T_n^{(4)} = 2^5 \sum_{j=1}^{n} U_j^{(1)}$, i.e. $T_3^{(4)} = 32 \cdot 15 = 480$.

Because of the information given in Theorem 10, the reader - if interested - could fairly readily construct a convolution table for $T_n^{(m)}$ corresponding to Table 1 for $U_n^{(m)}$.

Exploiting (2.1) and (5.1) together, we readily deduce ($m = 0$) the familiar relationship $T_n(x) = 2xU_n(x) - 2U_{n-1}(x)$ (see (1.9), $x = 1$), while $m = 1$ yields

$$T_n^{(1)}(x) = 4x^2 U_n^{(1)}(x) - 8xU_{n-1}^{(1)}(x) + 4U_{n-2}^{(1)}(x). \tag{5.3}$$

6. CHEBYSHEV AND PELL NUMBERS CONNECTED

A1. Pell $\longrightarrow$ Chebyshev $P \longrightarrow U$:

Pell number P_n are defined recursively [7, (1.1); $x = 1$] by

$$P_n = 2P_{n-1} + P_{n-2} (n \geq 2), P_0 = 0, P_1 = 1. \tag{6.1}$$

Recall that $U_0 = 0, U_1 = 1$ also, i.e., both P, U are of *Fibonacci-type*. Combine (6.1) and (1.7) ($x = 1$). Then these common initial conditions impose the summation relations recored in Table 2.

$$\begin{aligned}P_1 &= U_1\\ P_2 &= U_2\\ P_3 &= U_3 + 2U_1\\ P_4 &= U_4 + 4U_2\\ P_5 &= U_5 + 6U_3 + 6U_1\\ P_6 &= U_6 + 8U_4 + 16U_2\\ P_7 &= U_7 + 10U_5 + 30U_3 + 22U_1\\ P_8 &= U_8 + 12U_6 + 48U_4 + 68U_2\\ P_9 &= U_9 + 14U_7 + 70U_5 + 146U_3 + 90U_1\\ \cdots &\cdots \cdots\cdots\cdots\cdots\cdots\end{aligned} \tag{6.2}$$

Table 2. P_n in terms of U_n.

Hidden in this information is a pattern of formation which is empirically obtained (and readily verifiable). Let

$$u_{n,n-2k} = \text{the coefficient of } U_{n-2k} \text{ in the expansion of } P_n \text{ in Table 1.} \tag{6.3}$$

Then we discover the fundamental

Law of Formation A:

$$u_{n,n-2k} = u_{n-1,n-2k-1} + u_{n-1,n-2k+1} + u_{n-2,n-2k}. \tag{6.4}$$

For instance, for P_9 we have - follow the arrows in the "step" process in Table 2 -

$$u_{9,5}(=70) = u_{8,4} + u_{8,6} + u_{7,5}(= 48 + 12 + 10).$$

Always, $u_{n,n-4} = (n-4)u_{n,n-2} = 2(n-2)(n-4)$.

After appropriate computation using (2.2) with [4, (2.1)] and noting that $P_n^{(0)} \equiv P_9, P_0^{(9)} = U_0^{(9)} = 0$. We have, for example,

$$\begin{aligned}
P_9^{(0)} &= U_9^{(0)} + 14U_7^{(0)} + 70U_5^{(0)} + 146U_3^{(0)} + 90U_1^{(0)} \\
P_8^{(1)} &= U_8^{(1)} + 14U_6^{(1)} + 70U_4^{(1)} + 146U_2^{(1)} \\
P_7^{(2)} &= U_7^{(2)} + 14U_5^{(2)} + 70U_3^{(2)} + 146U_1^{(2)} \\
P_6^{(3)} &= U_6^{(3)} + 14U_4^{(3)} + 70U_2^{(3)} \\
P_5^{(4)} &= U_5^{(4)} + 14U_3^{(4)} + 70U_1^{(4)} \\
P_4^{(5)} &= U_4^{(5)} + 14U_2^{(5)} \\
P_3^{(6)} &= U_3^{(6)} + 14U_1^{(6)} \\
P_2^{(7)} &= U_2^{(7)} \\
P_1^{(8)} &= U_1^{(8)}.
\end{aligned} \tag{6.5}$$

This is a very revealing and appealing structural pattern with a regular contraction in the number of terms on the right-hand side.

As the pattern in (6.5) from $P_9^{(0)}, \cdots, P_0^{(9)}$ is typical, we may infer the general rule:

Rule of Convolution Transformation:

For two successive rows in (6.5) and for each U,

(a)

$$P_n^{(k)} \longrightarrow P_{n-1}^{(k+1)} \Rightarrow U_{n-2j}^{(k)} \longrightarrow U_{n-2j-1}^{(k+1)} \begin{cases} j = 0, 1, 2, \cdots, \frac{n}{2} \ (n \text{ even}) \\ j = 0, 1, 2, \cdots, \frac{n-1}{2} \ (n \text{ odd}), \end{cases} \tag{6.6}$$

(b) as k increases, the fixed pattern of coefficients of the U gradually contracts on the right-hand side.

A2. Chebyshev $\longrightarrow$ Pell $U \longrightarrow P$:

Manipulating the Chebyshev and Pell recurrences (1.7) and (6.1) as in **A**, we obtain the patterned behaviour of the relationships among the U's and the P's identical with those obtained in **A** except that the signs in even-numbered columns are - instead of +. Calculation yields

$$\begin{aligned} U_1 &= P_1 \\ U_2 &= P_2 \\ U_3 &= P_3 - 2P_1 \\ U_4 &= P_4 - 4P_2 \\ U_5 &= P_5 - 6P_3 + 6P_1 \\ U_6 &= P_6 - 8P_4 + 16P_2 \\ U_7 &= P_7 - 10P_5 + 30P_3 - 22P_1 \\ U_8 &= P_8 - 12P_6 + 48P_4 - 68P_2 \\ U_9 &= P_9 - 14P_7 + 70P_5 - 146P_3 + 90P_1 \\ \cdots &\ \cdots \cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{6.7}$$

Table 3. U_n in terms of P_n.

Next, let

$$p_{n,n-2k} = \text{the } \textit{absolute value} \text{ of the coefficient of } P_{n-2k} \text{ in the expansion in (6.2) of } U_n. \tag{6.8}$$

Then, the law of transformation embodied in (6.4) carries over to the

Law of Transformation | A |:

$$p_{n,n-2k} = p_{n-1,n-2k-1} + p_{n-1,n-2k+1} + p_{n-2,n-2k}. \tag{6.9}$$

So for example, $P_{10,4} = p_{9,3} + p_{9,5} + p_{8,4} (= 246)$.

***m*th Convolutions**:

Not unexpectedly, we again reach a system of expressions identical with that in (6.5) except for - signs instead of + signs in the even-numbered columns. Accordingly, for instance,

$$U_5^{(2)} = P_5^{(2)} - 10P_3^{(2)} + 30P_1^{(2)} (= 126),$$

$$U_7^{(2)} = P_7^{(2)} - 14P_5^{(2)} + 70P_3^{(2)} - 146P_1^{(2)} (= 532).$$

A law of convolution transformations corresponding to (6.6) is discernible.

Properties of $P_n^{(m)}$ and $U_n^{(m)}$ set out in A1 and A2 are special to these convolutions and do not carry over to other pairs of convolutions.

Efforts to obtain P_n as a compact closed summation form involving U's, and reversely, have been of no avail.

B. Pell-Lucas - Chebyshev $Q \leftrightarrow T$:

Pell-Lucas numbers Q_n are defined recursively [7; (1.2), $x = 1$] by

$$Q_n = 2Q_{n-1} + Q_{n-2}(n \geq 2), Q_0 = 2, Q_1 = 2. \tag{6.10}$$

Recall (1.11) that $T_0 = 2, T_1 = 2$ also, i.e., Q, T are of *Lucas type.*

B. $Q \longrightarrow T$:

Take (1.8), with $x = 1$, and (6.10) together to produce Table 4.

$$\begin{aligned}
Q_0 &= T_0 \\
Q_1 &= T_1 \\
Q_2 &= T_2 + 2T_0 \\
Q_3 &= T_3 + 6T_1 \\
Q_4 &= T_4 + 8T_2 + 8T_0 \\
Q_5 &= T_5 + 10T_3 + 30T_1 \\
Q_6 &= T_6 + 12T_4 + 48T_2 + 38T_0 \\
Q_7 &= T_7 + 14T_5 + 70T_3 + 154T_1 \\
Q_8 &= T_8 + 16T_6 + 96T_4 + 272T_2 + 192T_0 \\
Q_9 &= T_9 + 18T_7 + 126T_5 + 438T_3 + 810T_1 \\
Q_{10} &= T_{10} + 20T_8 + 160T_6 + 660T_4 + 1520T_2 + 1002T_0
\end{aligned} \tag{6.11}$$

Table 4. Q_n in terms of T_n.

Letting

$$t_{n,n-2k} = \text{the coefficient of } T_{n-2k} \text{ in the expansion of } Q_n \text{ in Table 4,} \tag{6.12}$$

we discover the

Law of Formation B:

$$t_{n,n-2k} = \delta q_{n-1,n-2k-1} + q_{n-1,n-2k+1} + q_{n-2,n-2k} \tag{6.13}$$

in which

$$\delta = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd}, n - 2k = 1. \end{cases}$$

For instance, $n = 9, k = 4$ yeilds $t_{9,1} = 2 \times 192 + 272 + 154 (= 810)$.

Observe that

$$\sum_{k=0}^{[\frac{n}{2}]} t_{n,n-2k} = \frac{1}{2} Q_n.$$

Convolutions $Q^{(1)} \longrightarrow T^{(1)}$:

We need

$$Q^{(1)}_{n+1} = 2Q^{(1)}_n + Q^{(1)}_{n-1} + 8P_{n+1}, Q^{(1)}_0 = 0[8; (5.4), x = 1; (5.1)], \tag{6.14}$$

$$T^{(1)}_{n+1} = 2T^{(1)}_n - T^{(1)}_{n-1}, T^{(1)}_0 = 0 \text{ (observation)}. \tag{6.15}$$

Then

$$\begin{aligned}
Q^{(1)}_1 &= T^{(1)}_1 && + 8(P^{(1)}_1 - P_1)\\
Q^{(1)}_2 &= T^{(1)}_2 && + 8(P^{(1)}_2 - P_2)\\
Q^{(1)}_3 &= T^{(1)}_3 + 2T^{(1)}_1 && + 8(P^{(1)}_3 - P_3)\\
Q^{(1)}_4 &= T^{(1)}_4 + 4T^{(1)}_2 && + 8(P^{(1)}_4 - P_4)\\
Q^{(1)}_5 &= T^{(1)}_5 + 6T^{(1)}_3 + 6^{(1)}_1 && + 8(P^{(1)}_5 - P_5)\\
Q^{(1)}_6 &= T^{(1)}_6 + 8T^{(1)}_4 + 16T^{(1)}_2 && + 8(P^{(1)}_6 - P_6)\\
Q^{(1)}_7 &= T^{(1)}_7 + 10T^{(1)}_5 + 30T^{(1)}_3 + 22T^{(1)}_1 && + 8(P^{(1)}_7 - P_7)\\
Q^{(1)}_8 &= T^{(1)}_8 + 12T^{(1)}_6 + 48T^{(1)}_4 + 68T^{(1)}_2 && + 8(P^{(1)}_8 - P_8)
\end{aligned} \tag{6.16}$$

Table 5. $Q^{(21)}_n$ in terms of $T^{(1)}_n$ and $P^{(1)}_n - P_n$.

Note that the pattern of coefficients of $T^{(1)}$ is identical with the pattern of coefficients of U_n in Table 2 (but not Table 4), i.e., Law of Formation B applies to the main (non-bracketed) portion of Table 5. [Question: Does a similar situation relate $T^{(1)}_n$ to $U^{(1)}_n$?]

B2. Chebyshev-Pell-Lucas $T \longrightarrow Q$:

Proceeding as in $B1 : Q \longrightarrow T$, we find the same pattern of formation as in (6.12) but with even-numbered columns distinguished by $-$ signs so that a table is readily constructible. For example

$$T_7 = Q_7 - 14Q_5 + 70Q_3 - 154Q_1 (= 2).$$

Convolutions $T^{(1)} \longrightarrow Q^{(1)}$:

These commence in an encouraging way in relation to Table 5, subject to change of signs in the non-bracketed portion, but soon deviate from it. For instance,

$$T^{(1)}_5 = Q^{(1)}_5 - 6Q^{(1)}_3 + 6Q^{(1)}_1 - 8 \text{ (a complicated expression involving } P\text{'s)}.$$

The sign proceeding 8 here is always $-$.

Faced with increasing difficulties, we reluctantly abandon further considerations of convolutions connecting Q and T.

C. Hybrid Transformations $P \leftrightarrow Q, U \leftrightarrow T, P \leftrightarrow T, Q \leftrightarrow U$:

Tables have been developed for these mixed Fibonacci and Lucas type polynomials i.e., for which the initial conditions are different. Much variety appears in these structures. Some patterns follow previous principles, some have no apparent underlying form. Fractions may be necessary, e.g., $P_1 = \frac{1}{2}Q_0(=1), U_1 = \frac{1}{2}T_0(=1)$. No convolution properties have been considered.

¿From the special nature of U (1.10) and T (1.11), the compositions of the tables are exceedingly simple, as expected.

A few random examples of the hybrid transformations are listed, merely for interest.

$$\begin{aligned} &U_7 = T_7 + T_5 + T_3 + \frac{1}{2}T_1, && T_9 = U_9 - U_7 \\ &U_6 = U_7 + 11U_5 + 36U_3 + 28U_1, && U_5 = Q_4 - 7Q_2 + 6\frac{1}{2}Q_0, \\ &P_7 = Q_6 - Q_4 + Q_2 - \frac{1}{2}Q_0, && Q_5 = 2P_5 + 6P_3 - 6P_1, \\ &T_8 = 2P_8 - 27P_6 + 121P_4 - 188P_2, && P_5 = 2T_3 + 12T_1 + \frac{1}{2}T_{-1}(T_{-1} = 2). \end{aligned}$$

7. CONCLUDING REMARKS

Negative Subscripts $R_{-n}, S_{-n}; c_{-n}, d_{-n}$:

Replace n by $-n$ in the definitions (1.1), (1.3) for $R_n^{(r,u)}(x)$, with $c_{n,k}^{(r,u)}$, and (1.4), (1.6) for $S_n^{(r,u)}(x)$, with $d_{n,k}^{(r,u)}$. Negative subscripts have already been introduced in [4, pp.189-191] to express $R_n^{(r,u)}(x)$ and $S_n^{(r,u)}(x)$ in terms of each other.

Steady calculations lead us to the empirically-derived Propositions 1 and 2 in which, by 1 we understand the non-r, non-u unit constant in $c_{n,k}^{(r,u)}$ and $d_{n,h}^{(r,u)}$. Notation is now reduced to a minimum, for simplicity.

Proposition 1:

$$\left.\begin{array}{r} R_n \longrightarrow R_{-n} \\ c_n \longrightarrow c_{-n} \end{array}\right\} \Rightarrow u \longrightarrow u, r \longrightarrow -r, 1 \longrightarrow u-1. \tag{7.1}$$

Proposition 2:

$$\left.\begin{array}{r} S_n \longrightarrow S_{-n} \\ d_n \longrightarrow d_{-n} \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} u \longrightarrow u, r \longrightarrow -r, 1 \longrightarrow -1+u & n \text{ even} \\ u \longrightarrow -u, r \longrightarrow r, 1 \longrightarrow 1-u & n \text{ odd} \end{array}\right. . \tag{7.2}$$

Examples:

(i) $R_3 = u + 3r + (3 + 3u + 4r)x + (4 + u + r)x^2 + x^3$ becomes by (7.1)
$R_{-3} = u - 3r + (-3 + 6u - 4r)x + (-4 + 5u - r)x^2 + (u-1)x^3$.

(ii) $S_4 = 17u + 12r + (12 + 18u + 14r)x + (14 + 7u + 6r)x^2 + (6 + r + u)x^3 + x^4$ becomes by (7.2)

$$\begin{aligned} S_{-4} = &17u - 12r + (-12 + 30u - 14r)x + (-14 + 21u - 6r)x^2 + (-6 + 7u - r)x^3 \\ &+ (-1+u)x^4. \end{aligned}$$

These calculations may be checked by means of (1.1) and (1.4) on allowing n to be negative.

Example (ii) means, for instance, that $d_{4,1} = 12 + 18u + 4r$ transforms to $d_{-4,1} = -12 + 30u - 14r$ by Proposition 2, n even.

What connection is there between $c_{n,k}^{(r,u)}$ and $d_{n,k}^{(r,u)}$ for negative n? Suffice it here to illustrate this extensive area of information with $R_3 = S_3 - 4S_1 + 2S_{-1}$ [4, p. 188] and $S_{-3} = -R_{-3} - 4R_{-1} - 2R_1$ [4, p. 189]. Comparison of coefficients of $x^k (k = 0, 1, 2, 3)$ on both sides in each case reveals that

$$\begin{aligned}
&c_{3,0} = d_{3,0} - 4d_{1,0} + 2d_{-1,0}, \quad c_{3,1} = d_{3,1} - 4d_{1,1} + 2d_{-1,1},\\
&c_{3,2} = d_{3,2}, c_{3,3} = d_{3,3}, \qquad \text{and}\\
&d_{-3,0} = -c_{-3,0} - 4c_{-1,0} - 2c_{1,0}, d_{-3,1} = -c_{-3,1} - 4c_{-1,1} - 2c_{1,1},\\
&d_{-3,2} = -c_{-3,2},\ d_{-3,3} = -c_{-3,3}.
\end{aligned}$$

For the general situation, which is very complicated, one may refer to [4, Theorem 7(a), (b), (c), (d), pp. 189-191].

Rising and Descending Diagonal Polynomials:

No attempt has been made to investigate rising and falling diagonal convolution polynomials for $T_n^{(m)}(x)$ and $U_n^{(m)}(x)$. However, rising and falling diagonal polynomials for $P_n(x), Q_n(x), T_n(x)$ and $U_n(x)$, occurring when $m = 0$, have been examined elsewhere in the literature, e.g. [2], and A.F. Horadam *The Fibonacci Quarterly* 16.1 (1978) pp. 33-36 and 18.1 (1980) pp. 3-8.

Strictures of time and space preclude a deeper, though desired, treatment of the objectives of this paper, stated in Section 1 (Motivation). Along the way, signposts have indicated directions for possible further analysis.

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