π IN TERMS OF ϕ

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ABSTRACT

In this paper, we derive some new formulas for π, similar to that of Bailey, Borwein and Plouffe. The distinctive feature of these new formulas is that π is expressed in terms of the powers of the reciprocal of the Golden Ratio ϕ.

In [3], with the aid of the powerful PSLQ algorithm [4, 6], David Bailey, Peter Borwein and Simon Plouffe discovered an amazing formula for π:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right). \tag{1}$$

This is a ground-breaking result because this formula can generate the nth base-16 digit of π without computing any prior digits, contrary to all previous algorithms for the nth digit of π. For introductions and generalizations, see, e.g., [1, 2, 5]; see also the lucid account in Hijab's book [10]. For a compendium of currently known results of BBP-type formulas, see Bailey's *A Compendium of BBP-Type Formulas for Mathematical Constants*, which is available at http://crd.lbl.gov/~dhbailey. See also [8].

In this paper, motivated by this beautiful result, we prove the following formulas. Denote the Golden Ratio by $\phi = (1+\sqrt{5})/2$. Then, we have

$$\pi = \frac{5\sqrt{2+\phi}}{2\,\phi} \sum_{n=0}^{\infty} \left(\frac{1}{2\phi}\right)^{5n} \left(\frac{1}{5n+1} + \frac{1}{2\phi^2(5n+2)} - \frac{1}{2^2\phi^3(5n+3)} - \frac{1}{2^3\phi^3(5n+4)} \right) \tag{2}$$

and

$$\pi = \frac{5\sqrt{2+\phi}}{2\,\phi^2} \sum_{n=0}^{\infty} \left(\frac{1}{\phi}\right)^{10n} \left(\frac{1}{10n+1} + \frac{1}{10n+2} + \frac{1}{\phi(10n+3)} + \frac{1}{\phi^3(10n+4)} \right.$$
$$\left. - \frac{1}{\phi^5(10n+6)} - \frac{1}{\phi^5(10n+7)} - \frac{1}{\phi^6(10n+8)} - \frac{1}{\phi^8(10n+9)} \right). \tag{3}$$

Proof of Formula 2: First, we observe that

$$\int_0^{1/(2\phi)} \frac{1}{1-\phi^{-1}x+x^2}\,dx = \frac{1}{5}\sqrt{\frac{2}{5+\sqrt{5}}}\,\pi. \tag{4}$$

Note that we have used in (4) the fact that

$$\tan\frac{\pi}{10} = \frac{\sqrt{5}-1}{\sqrt{2(5+\sqrt{5})}}.$$

Next, we define

$$A_1(x) := -1 - \phi^{-1}x + \phi^{-1}x^2 + x^3. \tag{5}$$

Observe that

$$x^5 - 1 = (1 - \phi^{-1}x + x^2)\, A_1(x). \tag{6}$$

By using (4)-(6), we have, with $a := 1/(2\phi)$,

$$\begin{aligned}\frac{\pi}{5\sqrt{2+\phi}} &= \int_0^a \frac{1}{1-\phi^{-1}x+x^2}\, dx \\ &= \int_0^a \frac{-A_1(x)}{1-x^5}\, dx \\ &= \int_0^a \frac{1+\phi^{-1}x-\phi^{-1}x^2-x^3}{1-x^5}\, dx.\end{aligned} \tag{7}$$

Following [3], we have, for fixed k,

$$\int_0^a \frac{x^{k-1}}{1-x^5}\, dx = \int_0^a \sum_{n=0}^{\infty} x^{k-1+5n}\, dx = \left(\frac{1}{2\,\phi}\right)^k \sum_{n=0}^{\infty} \left(\frac{1}{2\,\phi}\right)^{5n} \frac{1}{(5n+k)}. \tag{8}$$

By applying (8) to (7), we have

$$\frac{\pi}{5\sqrt{2+\phi}} = \sum_{n=0}^{\infty} \left(\frac{1}{2\,\phi}\right)^{5n} \left(\frac{1}{2\phi(5n+1)} + \frac{1}{2^2\phi^3(5n+2)} - \frac{1}{2^3\phi^4(5n+3)} - \frac{1}{2^4\phi^4(5n+4)}\right),$$

which is the same as (2).

Remark: Note that, by changing the upper limit of the integral in (4), i.e., $(1/2\phi) \to 1/\phi$, we have

$$\int_0^{1/\phi} \frac{1}{1-\phi^{-1}x+x^2}\, dx = \frac{1}{5}\sqrt{2-\frac{2}{\sqrt{5}}}\,\pi.$$

Hence, by the same tricks, one can show that

$$\pi = \frac{5\sqrt{2+\phi}}{2\,\phi} \sum_{n=0}^{\infty} \left(\frac{1}{\phi}\right)^{5n} \left(\frac{1}{5n+1} + \frac{1}{\phi^2(5n+2)} - \frac{1}{\phi^3(5n+3)} - \frac{1}{\phi^3(5n+4)}\right). \tag{9}$$

Proof of Formula 3: To this end, we observe that

$$\int_0^{1/\phi} \frac{1}{1-\phi\,x+x^2}\,dx = \frac{1}{5}\sqrt{2+\frac{2}{\sqrt{5}}}\,\pi. \tag{10}$$

Next, we define

$$A_2(x) := -1 - \phi\,x - \phi\,x^2 - x^3 + x^5 + \phi\,x^6 + \phi\,x^7 + x^8 \tag{11}$$

and observe that

$$x^{10} - 1 = (1 - \phi\,x + x^2)\,A_2(x). \tag{12}$$

By putting (10)-(12), we have, with $b := 1/\phi$,

$$\begin{aligned}
\frac{2\,\phi}{5\sqrt{2+\phi}}\,\pi &= \int_0^b \frac{1}{1-\phi\,x+x^2}\,dx \\
&= \int_0^b \frac{-A_2(x)}{1-x^{10}}\,dx \\
&= \int_0^b \frac{1+\phi\,x+\phi\,x^2+x^3-x^5-\phi\,x^6-\phi\,x^7-x^8}{1-x^{10}}\,dx.
\end{aligned}$$

By combining this with

$$\int_0^b \frac{x^{k-1}}{1-x^{10}}\,dx = \int_0^b \sum_{n=0}^{\infty} x^{k-1+10n}\,dx = \left(\frac{1}{\phi}\right)^k \sum_{n=0}^{\infty} \left(\frac{1}{\phi}\right)^{10n} \frac{1}{(10n+k)},$$

we obtain (3) in the same manner we obtained (2).

Remark: Again, consider the integral in (10) with a different upper limit ($1/\phi \to \phi/2$); we have

$$\int_0^{\phi/2} \frac{1}{1-\phi\,x+x^2}\,dx = \frac{3}{5}\sqrt{\frac{5+\sqrt{5}}{10}}\,\pi.$$

By the same tricks used to prove (3), we can show

$$\pi = \frac{5\sqrt{2+\phi}}{6}\sum_{n=0}^{\infty}\left(\frac{\phi}{2}\right)^{10n}\left(\frac{1}{10n+1}+\frac{\phi^2}{2(10n+2)}+\frac{\phi^3}{2^2(10n+3)}+\frac{\phi^3}{2^3(10n+4)}\right.$$
$$\left.-\frac{\phi^5}{2^5(10n+6)}-\frac{\phi^7}{2^6(10n+7)}-\frac{\phi^8}{2^7(10n+8)}-\frac{\phi^8}{2^8(10n+9)}\right).$$

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