ON THE SET OF REDUCED ϕ-PARTITIONS OF A POSITIVE INTEGER*

Jun Wang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P.R. China

Xin Wang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P.R. China

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ABSTRACT

Given a positive integer n, the sum $n = a_1 + \cdots + a_i$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_i \in \mathbb{N}$ is called a ϕ-partition if it satisfies $\phi(n) = \phi(a_1) + \cdots + \phi(a_i)$, where ϕ is Euler's totient function. And, a ϕ-partition is reduced if each of its summands is simple, where a simple number is known as 1 or a product of the first primes. In this note we will present a new algorithm to exhaust the set of all reduced ϕ-partitions of n.

1. INTRODUCTION

A partition of $n \in \mathbb{N}$, the set of all positive integers, is defined to be the sum $n = a_1 + \cdots + a_i$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_i \in \mathbb{N}$. In [1], Jones introduced an interesting partition: the sum $n = a_1 + \cdots + a_i$ is called a ϕ-partition if it satisfies $\phi(n) = \phi(a_1) + \cdots + \phi(a_i)$, where ϕ is Euler's totient function. Furthermore, a ϕ-partition is reduced if each of its summands is simple, where a simple number is known as 1 or a product of the first primes. More precisely, let p_i denote the i-th prime and define $A_0 = 1$ and $A_i = \prod_{j=1}^{i} p_j$, which is the i-th simple number. Jones proved that every simple number has the only trivial ϕ-partition $A_i = A_i$, and each non-simple number n has a nontrivial ϕ-partition as follows: Let p and q denote distinct primes. Then

(I) $n = \underbrace{p^{\alpha-1}t + \cdots + p^{\alpha-1}t}_{p}$ if $n = p^\alpha t$ for $\alpha > 1$ and $p \nmid t$,

(II) $n = \underbrace{j + \cdots + j}_{p-q} + qj$ if $n = pj$ where p and q do not divide j and $q < p$.

This gives algorithms from which we can obtain at least one reduced ϕ-partition of any non-simple number. In fact, we can regard a reduced ϕ-partition of n as a solution of the following system of equations in $(x_0, x_1, \dots)$:

$$\begin{cases} n &= x_0 + x_1 A_1 \quad + x_2 A_2 + \dots \\ \phi(n) &= x_0 + x_1 \phi(A_1) \quad + x_2 \phi(A_2) + \dots \end{cases} \tag{1.1}$$

such that x_j's are non-negative integers.

Let $S(n)$ and $S^+(n)$ denote the sets of all integer and nonnegative integer solutions of (1.1), respectively. A positive integer n is called semisimple if it has exactly one reduced ϕ-partition, that is, $|S^+(n)| = 1$.

In [3], a complete characterization of semisimple integers was given (cf. [2]):

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Theorem 1.1: *Let n be a nonsimple integer. Then n is semisimple if and only if n is a prime, or $n = 3^2$, or $n = aq_1 \cdots q_k A_i$ with $a(q_1 - p_{i+1}) \cdots (q_k - p_{i+1}) < p_{i+1}$, where $i \geq 1, k \geq 0, q_1 > q_2 > \cdots > q_k > p_{i+1}$ are primes and a is a positive integer.*

In [3] it also asked for the set $S^+(n)$ for any non-semisimple number n. In this note, we present a new algorithm to exhaust the set.

2. ALGORITHM

By S we denote the set of all sequences $\boldsymbol{a} = (a_0, a_1, \dots)$ such that a_j's are integers and all but finite many of them are zero. (Here, a bold letter always denotes an element of S.) For convenience, we omit its terminal 0's, for example, we write $(a_0, \dots, a_i, 0, \dots) = (a_0, \dots, a_i)$.

Given a subset T of S, let $\hat{T}$ denote the subset of T consisting of all $\boldsymbol{a} = (a_0, a_1, \dots)$'s such that $a_r \geq 0$ for $r \geq 1$; and let T^+ denote the set of all nonnegative integer sequences in T. Clearly, $\hat{S}(n) \subset \hat{S}$ and $S^+(n) \subset S^+$. Given an $\boldsymbol{a} = (a_0, a_1, \dots, a_j, \dots) \in S$, let $S_j(\boldsymbol{a})$ denote the set of all integer solutions of the following system of equations:

$$\begin{cases} \sum_{r=0}^{j} a_r A_r &= \sum_{r=0}^{j} x_r A_r + A_{j+1} \\ \sum_{r=0}^{j} a_r \phi(A_r) &= \sum_{r=0}^{j} x_r \phi(A_r) + \phi(A_{j+1}). \end{cases} \tag{2.2}$$

Define a linear order "$\preceq$" on S to be the "right" lexicographic order, that is, $\boldsymbol{a} \prec \boldsymbol{b}$ if $a_i < b_i$, for some $i \geq 0$, and $a_{i+j} = b_{i+j}$ for all $j > 0$. Given a subset $T \subset S$ and $\boldsymbol{a}, \boldsymbol{b} \in T$, we say $\boldsymbol{a}$ and $\boldsymbol{b}$ are adjacent in T if there is no $\boldsymbol{c} \in T$ with $\boldsymbol{a} \prec \boldsymbol{c} \prec \boldsymbol{b}$.

For exhausting the set $S^+(n)$ we proceed to give a new algorithm by solving the system (2.2).

We define an operator $\mathfrak{E}$ on $S^+(n)$ by $\mathfrak{E}(\boldsymbol{a}) = \boldsymbol{a}$ if $S_j^+(\boldsymbol{a}) = \emptyset$ for all $j \geq 1$; otherwise,

$$\mathfrak{E}(\boldsymbol{a}) = (y_0, y_1, \dots, y_j, a_{j+1} + 1, a_{j+2}, \dots), \tag{2.3}$$

where j is the least positive index with $S_j^+(\boldsymbol{a}) \neq \emptyset$ and $\boldsymbol{y} = (y_0, y_1, \dots, y_j)$ is the minimum element of $S_j^+(\boldsymbol{a})$ in order $\preceq$. Clearly, $\boldsymbol{a} \in S^+(n)$ implies that $\mathfrak{E}(\boldsymbol{a}) \in S^+(n)$ and $\boldsymbol{a} \preceq \mathfrak{E}(\boldsymbol{a})$.

Furthermore, define $\mathfrak{E}^0$ to be the identity map, $\mathfrak{E}^{-1}$ to be the inverse of $\mathfrak{E}$, that is, $\mathfrak{E}^{-1}(\boldsymbol{b}) = \boldsymbol{a}$ if $\boldsymbol{b} = \mathfrak{E}(\boldsymbol{a})$. For an integer t we inductively define the operator $\mathfrak{E}^t$ by $\mathfrak{E}^t(\boldsymbol{a}) = \mathfrak{E}(\mathfrak{E}^{t-1}(\boldsymbol{a}))$ for $\boldsymbol{a} \in S^+(n)$.

It is evident that if $\boldsymbol{a}$ is maximal in $(S^+(n), \preceq)$, then $\mathfrak{E}(\boldsymbol{a}) = \boldsymbol{a}$, in other words, $S_j^+(\boldsymbol{a}) = \emptyset$ for all $j \geq 1$. We now characterize the maximum element of $S^+(n)$. To do this, we introduce some notations.

For $j \geq 1$, write $\Gamma_j = A_j - \phi(A_j)$. It is easy to see that $(p_{j+1} + 1)\Gamma_j > \Gamma_{j+1} > p_{j+1}\Gamma_j$. (See [3] Lemma 3.)

Lemma 2.1: *Let $\boldsymbol{a} = (a_0, a_1, \dots)$ be in $S^+(n)$. Then $\mathfrak{E}(\boldsymbol{a}) \succ \boldsymbol{a}$ if and only if there is an index $j > 0$ such that*

$$\sum_{r=1}^{j} a_r \Gamma_r \geq \Gamma_{j+1}. \tag{2.4}$$

Proof: Keep the notation of $\mathfrak{E}(\boldsymbol{a})$ as in (2.3). Suppose $\mathfrak{E}(\boldsymbol{a}) \succ \boldsymbol{a}$. Then $\boldsymbol{y} = (y_0, y_1, \dots, y_j)$ is a nonnegative integer solution of (2.2). Subtracting the second equation from the first in (2.2) and substituting by $\boldsymbol{y}$, we get

$$\sum_{r=1}^{j} a_r \Gamma_r = \sum_{r=1}^{j} y_r \Gamma_r + \Gamma_{j+1} \geq \Gamma_{j+1},$$

as desired.

Conversely, if there is a $j \geq 1$ satisfying (2.4), we may suppose that this j is the least index with this property. Note that $\Gamma_1 = 1$, from which we see that there are nonnegative integers $x_1, \dots, x_j$ such that

$$\sum_{r=1}^{j} a_r \Gamma_r - \Gamma_{j+1} = \sum_{r=1}^{j} x_r \Gamma_r. \tag{2.5}$$

It is easy to check that $\boldsymbol{x} = (x_0, x_1, \dots, x_j) \in \hat{S}_j(\boldsymbol{a})$, where x_0 is given by

$$x_0 - a_0 = \sum_{r=1}^{j} a_r A_r - A_{j+1} - \sum_{r=1}^{j} x_r A_r = \sum_{r=1}^{j} a_r \phi(A_r) - \phi(A_{j+1}) - \sum_{r=1}^{j} x_r \phi(A_r).$$

Therefore, in order to complete the proof it suffices to prove that $z_0 - a_0 \geq 0$ for a $\boldsymbol{z} \in S_j^+(\boldsymbol{a})$.

Let $\boldsymbol{z} = (z_0, z_1, \dots, z_j)$ be the maximum element of $(\hat{S}_j(\boldsymbol{a}), \preceq)$. If $j = 1$, from $a_1\Gamma_1 = \Gamma_2 + z_1\Gamma_1$, i.e., $a_1 - z_1 = A_2 - \phi(A_2) = 4$, it follows that $z_0 - a_0 = (a_1 - z_1)p_1 - A_2 = 2 > 0$.

Assume now $j > 1$. By induction we may suppose that $\sum_{r=1}^{i-1} z_r \Gamma_r < \Gamma_i$ holds for each $1 \leq i < j$ because $\boldsymbol{z}$ is maximal in $(\hat{S}_j(\boldsymbol{a}), \preceq)$. In particular, $z_r < p_{r+1} + 1$ for $r < j$. By the minimality of j we may assume that $a_r < p_{r+1} + 1$ for $r < j$. From the minimality of j we can also see that $a_j > z_j$. Let i be the index such that $a_r \geq z_r$ for $i \leq r \leq j$ and $a_{i-1} < z_{i-1}$, where $1 \leq i \leq j$, and put $a'_r = a_r - z_r$ for $1 \leq r \leq j$. Then $|a'_r| < p_{r+1} + 1$ for $r < i$ and $\sum_{r=i}^{j} a'_r \Gamma_r \geq \Gamma_{j+1}$ (otherwise, $i > 2$ and $\sum_{r=1}^{i-2} a_r \Gamma_r > (z_{i-1} - a_{i-1})\Gamma_{i-1} \geq \Gamma_{i-1}$, which contradicts the choice of j). Set $\sigma_r = \frac{\Gamma_r}{A_r} = 1 - \frac{\phi(A_r)}{A_r}$. Then $\sigma_r < \sigma_{j+1}$ for $r < j + 1$. We thus have

$$\sum_{r=i}^{j} a'_r \Gamma_r = \sum_{r=i}^{j} a'_r A_r \sigma_r \geq \Gamma_{j+1} = A_{j+1}\sigma_{j+1},$$

which implies that $\sum_{r=i}^{j} a'_r A_r > A_{j+1}$.

Write $\sum_{r=i}^{j} a'_r A_r - A_{j+1} = tA_i$, where t is a positive integer. If $t \geq 2$, taking account of $|a'_r| \leq p_{r+1}$ for $r < i$, we then have

$$z_0 - a_0 = \sum_{r=1}^{j} a'_r A_r - A_{j+1} \geq 2A_i - \sum_{r=1}^{i-1} p_{r+1} A_r$$

$$= A_i - \sum_{r=2}^{i-1} A_r > 0,$$

which yields $S_j^+(\boldsymbol{a}) \neq \emptyset$. If $t = 1$, then $b_iA_i + \cdots + b_jA_j = A_{j+1}$, where $b_i = a_i' - 1$ and $b_r = a_r'$ for $i < r \leq j$. From the above case it is easy to see that $p_{j+1} - 1 \leq b_j \leq p_{j+1}$. If $b_j = p_{j+1}$, then $b_i = \cdots = b_{j-1} = 0$; if $b_j = p_{j+1} - 1$, then $b_iA_i + \cdots + b_{j-1}A_{j-1} = A_j$. Therefore, there is an s with $i \leq s \leq j$ such that $b_s = p_{s+1}$ and $b_r = 0$ if $r < s$ and $b_r = p_{r+1} - 1$ if $r > s$. Note that $(p_{j+1} - 1)\phi(A_j) = \phi(A_{j+1})$ and $z_r \leq p_{r+1}$ for $r < i$. Thus,

$$\begin{aligned} z_0 - a_0 &= \sum_{r=1}^{j} a_r\phi(A_r) - \phi(A_{j+1}) - \sum_{r=1}^{j} z_r\phi(A_r) \\ &\geq \phi(A_i) + \sum_{r=s}^{j} b_r\phi(A_r) - \phi(A_{j+1}) - \sum_{r=1}^{i-1} p_{r+1}\phi(A_r) \\ &\geq 2\phi(A_i) - \sum_{r=1}^{i-1} (\phi(A_{r+1}) + \phi(A_r)) \\ &> \phi(A_i) + 1 - 2\sum_{r=1}^{i-1} \phi(A_r) > 0. \end{aligned}$$

We still have $\mathcal{S}_j^+(\boldsymbol{a}) \neq \emptyset$. Therefore, (2.3) is specified as

$$\mathfrak{E}(\boldsymbol{a}) = (y_0, y_1, \ldots, y_j, a_{j+1} + 1, a_{j+2} \ldots), \tag{2.6}$$

where $(y_0, y_1, \ldots, y_j)$ is the minimum element of $S_j^+(\boldsymbol{a})$. □

¿From definition we can immediately obtain the following proposition, which characterizes the adjacency relation on $S^+(n)$, hence whole $S^+(n)$ can be obtained.

Theorem 2.2: *Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are in $S^+(n)$ with $\boldsymbol{a} \prec \boldsymbol{b}$. Then $\boldsymbol{a}$ and $\boldsymbol{b}$ are adjacent in $S^+(n)$ if and only if $\boldsymbol{b} = \mathfrak{E}(\boldsymbol{a})$. Thus,*

$$S^+(n) = \{\mathfrak{E}^t(\boldsymbol{a}) | t \in \mathbb{Z}, \boldsymbol{a} \textit{ is that one obtained by Algorithms I and II}\}.$$

3. CONCLUDING REMARKS

It can be seen from Theorem 1.1 that all odd integers but prime numbers and 3^2 are non-semisimple, while $(p_{i+1} - 1)A_i$ and $p_{i+2}A_i$ are semisimple for all $i \geq 1$. With the notation in Theorem 1.1 for $a = 1$ and $k \geq 2$, the smallest semisimple number is $p_9 \times p_8 \times A_6 = 23 \times 19 \times 13 \times 11 \times 7 \times 5 \times 3 \times 2$. For $p_8 \times p_7 \times A_5 = 746130$ we list the elements of $S^+(746130)$ as follows:

$(0, 0, 0, 0, 0, 24, 6, 1)$, $(270, 270, 90, 18, 2, 10, 7, 1)$, $(404, 2, 157, 18, 2, 10, 7, 1)$,
$(456, 54, 1, 44, 2, 10, 7, 1)$, $(482, 2, 14, 44, 2, 10, 7, 1)$, $(486, 6, 2, 46, 2, 10, 7, 1)$,
$(488, 2, 3, 46, 2, 10, 7, 1)$, $(518, 32, 13, 6, 7, 10, 7, 1)$, $(534, 0, 21, 6, 7, 10, 7, 1)$,
$(540, 6, 3, 9, 7, 10, 7, 1)$, $(542, 2, 4, 9, 7, 10, 7, 1)$, $(548, 8, 6, 1, 8, 10, 7, 1)$,
$(552, 0, 8, 1, 8, 10, 7, 1)$, $(554, 2, 2, 2, 8, 10, 7, 1)$.

We have seen that it is easier to determine an image of $\mathfrak{E}$ than that of $\mathfrak{E}^{-1}$. Therefore, in order to exhaust the set $S^+(n)$, it would be interesting to find the minimum element of $S^+(n)$ for any non-ϕ-semisimple number n. It is not difficult to verify that in the example above, the partition obtained by Algorithms (I) and (II) is just the minimum element of $S^+(746130)$. We guess that it is always the case for all non-semisimple numbers.

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