APPLICATIONS OF WARING'S FORMULA TO SOME IDENTITIES OF CHEBYSHEV POLYNOMIALS

Jiang Zeng

Institut Camille Jordan, Université Claude Bernard (Lyon I), 69622 Villeurbanne Cedex, France
e-mail: zeng@math.univ-lyon1.fr

Jin Zhou

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, People's Republic of China
e-mail: jinjinzhou@yahoo.com

(Submitted November 2003)

ABSTRACT

Some identities of Chebyshev polynomials are deduced from Waring's formula on symmetric functions. In particular, these formulae generalize some recent results of Grabner and Prodinger.

1. INTRODUCTION

Given a set of variables $X = \{x_1, x_2, \dots\}$, the kth ($k \geq 0$) *elementary symmetric polynomial* $e_k(X)$ is defined by $e_0(X) = 1$,

$$e_k(X) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}, \quad \text{for} \quad k \geqslant 1,$$

and the kth ($k \geq 0$) *power sum symmetric polynomial* $p_k(X)$ is defined by $p_0(X) = 1$,

$$p_k(X) = \sum_i x_i^k, \quad \text{for} \quad k \geqslant 1.$$

Let $\lambda = 1^{m_1} 2^{m_2} \dots$ be a partition of n, i.e., $m_1 1 + m_2 2 + \dots + m_n n = n$, where $m_i \geq 0$ for $i = 1, 2, \dots n$. Set $l(\lambda) = m_1 + m_2 + \dots + m_n$. According to the *fundamental theorem of symmetric polynomials*, any symmetric polynomial can be written uniquely as a polynomial of elementary symmetric polynomials $e_i(X)$ ($i \geq 0$). In particular, for the power sum $p_k(x)$, the corresponding formula is usually attributed to Waring [1, 4] and reads as follows:

$$p_k(X) = \sum_\lambda (-1)^{k - l(\lambda)} \frac{k(l(\lambda) - 1)!}{\prod_i m_i!} e_1(X)^{m_1} e_2(X)^{m_2} \dots, \tag{1}$$

where the sum is over all the partitions $\lambda = 1^{m_1} 2^{m_2} \dots$ of k.

In a recent paper [3] Grabner and Prodinger proved some identities about Chebyshev polynomials using generating functions, the aim of this paper is to show that Waring's formula provides a natural generalization of such kind of identities.

Let U_n and V_n be two sequences defined by the following recurrence relations:

$$U_n = pU_{n-1} - U_{n-2}, \quad U_0 = 0, U_1 = 1, \tag{2}$$

$$V_n = pV_{n-1} - V_{n-2}, \quad V_0 = 2, V_1 = p. \tag{3}$$

Hence U_n and V_n are rescaled versions of the second and first kind of Chebyshev polynomials $U_n(x)$ and $T_n(x)$, respectively:

$$U_{n+1} = U_n(p/2), \quad V_n = 2T_n(x).$$

Theorem 1: *For integers $m, n \geq 0$, let $W_n = aU_n + bV_n$ and $\Omega = a^2 + 4b^2 - b^2p^2$. Then the following identity holds*

$$W_n^{2k} + W_{n+m}^{2k} = \sum_{r=0}^{k} \theta_{k,r}(m)\Omega^{k-r} W_n^r W_{n+m}^r, \tag{4}$$

where

$$\theta_{k,r}(m) = \sum_{0 \leqslant 2j \leqslant k} (-1)^j \frac{k(k-j-1)!}{j!(k-r)!(r-2j)!} V_m^{r-2j} U_m^{2k-2r}.$$

Note that the identities of Grabner and Prodinger [3] correspond to the $m = 1$ and implicitly $m = 2$ cases of Theorem 1 (cf. Section 3).

2. PROOF OF THEOREM 1

We first check the $k = 1$ case of (4):

$$W_n^2 + W_{n+m}^2 = V_m W_n W_{n+m} + U_m^2 \Omega. \tag{5}$$

Set $\alpha = (p + \sqrt{p^2 - 4})/2$ and $\beta = (p - \sqrt{p^2 - 4})/2$ then it is easy to see that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n,$$

it follows that

$$W_n = aU_n + bV_n = A\alpha^n + B\beta^n,$$

where $A = b + a/(\alpha - \beta)$ and $B = b - a/(\alpha - \beta)$. Therefore

$$V_m W_n W_{n+m} + U_m^2 \Omega = (\alpha^m + \beta^m)(A\alpha^n + B\beta^n)(A\alpha^{n+m} + B\beta^{n+m})$$
$$+ \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right)^2 (a^2 + 4b^2 - b^2p^2),$$

which is readily seen to be equal to $W_n^2 + W_{n+m}^2$.

Next we take the alphabet $X = \{W_n^2, W_{n+m}^2\}$, then the left-hand side of (4) is the power sum $p_k(X)$. On the other hand, since

$$e_1(X) = W_n^2 + W_{n+m}^2, \quad e_2(X) = W_n^2 W_{n+m}^2, \quad e_i(X) = 0 \quad \text{if} \quad i \geqslant 3,$$

the summation at the right-hand side of (1) reduces to the partitions $\lambda = (1^{k-2j}\, 2^j)$, with $j \geq 0$. Now, using (5) Waring's formula (1) infers that

$$\begin{aligned} W_n^{2k} &+ W_{n+m}^{2k} \\ &= \sum_{0\leqslant 2j\leqslant k} (-1)^j \frac{k(k-j-1)!}{j!(k-2j)!} (V_m W_n W_{n+m} + U_m^2 \Omega)^{k-2j} (W_n^2 W_{n+m}^2)^j \\ &= \sum_{0\leqslant 2j\leqslant k} \sum_{i=0}^{k-2j} (-1)^j \frac{k(k-j-1)!}{j!i!(k-2j-i)!} V_m^{k-2j-i} U_m^{2i} \Omega^i (W_n W_{n+m})^{k-i} \end{aligned}$$

Setting $k-i=r$ and exchanging the order of summations yields (4). □

3. SOME SPECIAL CASES

When $m=1$ or 2, as $U_1=1$, $V_1=p$ and $U_2=p$, $V_2=p^2-2$ the coefficient $\theta_{k,r}(r)$ of Theorem 1 is much simpler.

Corollary 1: *We have*

$$\theta_{k,r}(1) = \sum_{0\leqslant 2j\leqslant r} (-1)^j \frac{k(k-1-j)!}{(k-r)!j!(r-2j)!} p^{r-2j}, \tag{6}$$

$$\theta_{k,r}(2) = \sum_{0\leqslant 2j\leqslant k} (-1)^j \frac{k(k-j-1)!}{j!(k-r)!(r-2j)!} (p^2-2)^{r-2j} p^{2k-2r}. \tag{7}$$

We notice that (6) is exactly the formula given by Grabner and Prodinger [3] for $\theta_{k,r}(1)$, while for $\theta_{k,r}(2)$ they give a more involved formula than (7) as follows:

Corollary 2: (Grabner and Prodinger [3]) There holds

$$\theta_{k,r}(2) = \sum_{0\leqslant \lambda\leqslant k} (-1)^\lambda p^{2k-2\lambda} \frac{k(k-\lfloor\frac{\lambda}{2}\rfloor-1)!2^{\lceil\frac{\lambda}{2}\rceil}}{(k-r)!\lambda!(r-\lambda)!} \prod_{i=0}^{\lfloor\frac{\lambda}{2}\rfloor-1} (2k-2\left\lceil\frac{\lambda}{2}\right\rceil-1-2i). \tag{8}$$

In order to identify (7) and (8), we need the following identity.

Lemma 2: *We have*

$$\begin{aligned} &\sum_{i=0}^{j/2} (-1)^i \frac{(k-i-1)!2^{j-2i}}{(j-2i)!i!} \\ &\qquad = \frac{(k-\lfloor j/2\rfloor-1)!}{j!} 2^{\lceil j/2\rceil} \prod_{i=0}^{\lfloor j/2\rfloor-1} (2k-2\lceil j/2\rceil-1-2i). \end{aligned} \tag{9}$$

Proof: For $n \geq 0$ let $(a)_n = a(a+1)\dots(a+n-1)$, then the Chu-Vandermonde formula [2, p. 212] reads:

$$ {}_2F_1(-n, a; c; 1) := \sum_{k \geqslant 0} \frac{(-n)_k (a)_k}{(c)_k k!} = \frac{(c-a)_n}{(c)_n}. \tag{10} $$

Note that $n! = (1)_n$, so using the simple transformation formulae:

$$ (a)_{2n} = \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n 2^{2n}, \quad (a)_{2n+1} = \left(\frac{a}{2}\right)_{n+1} \left(\frac{a+1}{2}\right)_n 2^{2n+1}, $$

and

$$ (a)_{N-n} = \frac{(a)_N}{(a+N-n)_n} = (-1)^n \frac{(a)_N}{(-a-N+1)_n}, $$

we can rewrite the left-hand side of identity (9) as follows:

$$ \begin{cases} \frac{(k-1)!}{(\frac{1}{2})_m (1)_m} \, {}_2F_1(-m, -m+\frac{1}{2}; -k+1; 1) & \text{if } j = 2m, \\ \\ \frac{(k-1)!}{(\frac{1}{2})_{m+1} (1)_m} \, {}_2F_1(-m, -m-\frac{1}{2}; -k+1; 1) & \text{if } j = 2m+1, \end{cases} $$

which is clearly equal to the right-hand side of (9) in view of (10). □

Now, expanding the right-hand side of (7) by binomial formula yields

$$ \sum_{0 \leqslant 2j \leqslant k} (-1)^j \frac{k(k-j-1)!}{j!(k-r)!(r-2j)!} \sum_{i=0}^{r-2j} \binom{r-2j}{i} p^{2i} (-2)^{r-2j-i} p^{2k-2r}. $$

Writing $\lambda = r - i$, so $\lambda \leq r \leq k$, and exchanging the order of summations, the above quantity becomes

$$ \sum_{0 \leqslant \lambda \leqslant k} (-1)^\lambda p^{2k-2\lambda} \frac{k}{(k-r)!(r-\lambda)!} \sum_{0 \leqslant j \leqslant k/2} (-1)^j \frac{(k-j-1)! 2^{\lambda-2j}}{(\lambda-2j)! j!}, $$

which yields (8) by applying Lemma 2.

REFERENCES

[1] Frédéric Jouhet and Jiang Zeng. "Généralisation de Formules de Waring." Séminaire Lotharingien de Combinatorie, B44g (2000), pp 9.

[2] Ronald L. Graham, Donald E. Knuth and Oren Patashnik. *Concrete Mathematics*, Addion-Wesley Pubilshing Co. 1989.

[3] Peter J. Grabner and Helmut Prodinger. "Some Identities for Chebyshev Polynomials." *Portugalia Mathematicae* **59** (2002): 311-314.

[4] P. A. MacMahon. *Combinatory analysis*, Chelsea Publishing Co., New York, 1960.

AMS Classification Numbers: 11B39, 33C05, 05E05

✠ ✠ ✠