FIFTH ROOTS OF FIBONACCI FRACTIONS

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ABSTRACT

We prove that when n is odd, the continued fraction expansion of $\sqrt[5]{\frac{F_{n+5}}{F_n}}$ begins with a string of 1's, followed by $F_{2n+5}+2$, and that when n is even, the expansion begins with a string of 1's, then a 2, then $F_{2n+5}-4$.

1. INTRODUCTION

Let ϕ be the golden ratio $\frac{1+\sqrt{5}}{2}$, and let $\bar{\phi}$ be its conjugate $\frac{1-\sqrt{5}}{2}$. According to Binet's formula, the n^{th} Fibonacci number F_n is given by

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}.$$

Since $|\bar{\phi}| < 1$, it follows that for any $k \in \mathbf{N}$,

$$\lim_{n\to\infty} \sqrt[k]{\frac{F_{n+k}}{F_n}} = \phi.$$

The continued fraction expansion of ϕ is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}},$$

which we will denote in the more compact notation $[1, 1, 1, \cdots]$. One might then expect the expansion of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$ to begin with a string of 1's. Indeed, as we show in Proposition 1, such expansions begin with at least n ones for any $k \in \mathbf{N}$.

When $k = 1$, this is, in fact, all one sees. That is, the continued fraction expansion of $\frac{F_{n+1}}{F_n}$ consists of a sequence of n 1's, as one can easily show by induction. When $k = 2$, one can show that the continued fraction expansion is periodic, consisting of repeating blocks of the sequence $\{1, 1, \cdots, 1, 2\}$, with $(n-1)$ 1's, after the initial 1. But for $k \geq 3$, since $\sqrt[3]{\frac{F_{n+3}}{F_n}}$ is neither rational nor quadratic irrational, it follows by a theorem due to Lagrange ([3, [p. 233]) that all expansions are non-repeating and non-terminating.

In fact, for $k \geq 3$, one observes a seemingly random sequence after the initial string of 1's. For example, when $k = 3$ and $n = 3$, the expansion begins

$$[1, 1, 1, 2, 2, 1, 3, 2, 3, 1, \cdots].$$

When $k = 4$ and $n = 4$, the expansion begins

$$[1, 1, 1, 1, 2, 9, 1, 1, 6, 2, \cdots].$$

But something remarkable happens when $k = 5$. The first few terms of the continued fraction expansions for $k = 5$ and $n = 1$ through $n = 6$ are listed below:

$$[1, 1, 1, 15, 2, 2, \cdots]$$
$$[1, 1, 2, 30, 2, 3, \cdots]$$
$$[1, 1, 1, 1, 1, 91, 2, 48, \cdots]$$
$$[1, 1, 1, 1, 2, 229, 2, 12, \cdots]$$
$$[1, 1, 1, 1, 1, 1, 1, 612, 1, 1, \cdots]$$
$$[1, 1, 1, 1, 1, 1, 2, 1593, 2, 18, \cdots]$$

Ardent Fibonacci enthusiasts will have observed immediately that the sequence of large numbers one sees above,

$$\{15, 30, 91, 229, 612, 1593, \cdots\}$$

is related to the Fibonacci sequence itself. Indeed,

$$15 = F_7 + 2,$$
$$30 = F_9 - 4,$$
$$91 = F_{11} + 2,$$
$$229 = F_{13} - 4,$$
$$612 = F_{15} + 2,$$
$$1593 = F_{17} - 4.$$

This curious observation is the main result of this paper.

Theorem 1: *Using the above notation for continued fractions, we have*

$$\sqrt[5]{\frac{F_{n+5}}{F_n}} = \begin{cases} [\overbrace{1, 1, \cdots, 1}^{n+2 \text{ 1's}}, F_{2n+5} + 2, \cdots] & n \text{ odd} \\ [\overbrace{1, 1, \cdots, 1}^{n \text{ 1's}}, 2, F_{2n+5} - 4, \cdots] & n \text{ even.} \end{cases}$$

2. PRELIMINARIES

One preliminary observation will be helpful throughout.

Lemma 1: *Suppose a and m are positive integers. Then*

$$[\overbrace{1,1,\cdots,1}^{m\ 1\text{'s}},a] = \frac{F_{m+1}a+F_m}{F_m a+F_{m-1}},$$

and

$$[\overbrace{1,1,\cdots,1}^{m\ 1\text{'s}},2,a] = \frac{F_{m+3}a+F_{m+1}}{F_{m+2}a+F_m}.$$

Proof: By induction on m. □

We will use the following lemma, which is standard in continued fraction theory, to obtain information about the truncations of continued fraction expansions.

Lemma 2: *Let α be any positive real number. If m is odd, then the m^{th} truncation of the continued fraction expansion of α is $[a_1,a_2,\cdots,a_m]$ if and only if*

$$[a_1,a_2,\cdots,a_m] \le \alpha < [a_1,a_2,\cdots,a_m+1].$$

If m is even, the same statement holds provided the inequalities are reversed.

Proof: By subtracting a_1 and then inverting, or inverting and then adding a_1, we see that

$$[a_1,a_2,\cdots,a_m] \le \alpha < [a_1,a_2,\cdots,a_m+1]$$

if and only if

$$[a_2,a_3,\cdots,a_m] \ge \frac{1}{\alpha-a_1} > [a_2,a_3,\cdots,a_m+1].$$

By induction, the second inequality holds if and only if the $(m-1)^{\text{st}}$ trunctation of $\frac{1}{\alpha-a_1}$ is $[a_2,a_3,\cdots,a_m]$. But this holds if and only if the m^{th} truncation of α is $[a_1,a_2,\cdots,a_m]$. □

We can use this lemma to prove that the first n terms in the continued fraction expansion of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$ are all 1's. The proof will be easier with the following lemma in hand as well.

Lemma 3: *Let k be any nonnegative integer and suppose n is odd. Then*

$$F_{n+1}^k \le F_{n+k}F_n^{k-1}$$

and

$$F_{n+1}^k F_{n+k} \le F_{n+2}^k F_n.$$

If n is even, the opposite inequalities hold.

Proof: First, suppose n is odd. When $k=0$, we have equality in both cases. When $k\ge 1$, we suppose the result holds for $k-1$, so

$$F_{n+1}^{k-1} \le F_{n+k-1}F_n^{k-2}$$

and

$$F_{n+1}^{k-1}F_{n+k-1} \le F_{n+2}^{k-1}F_n.$$

These imply

$$F_{n+1}^{k} \leq F_{n+k-1}F_{n+1}F_{n}^{k-2} \tag{1}$$

and

$$F_{n+2}F_{n+1}^{k-1}F_{n+k-1} \leq F_{n+2}^{k}F_{n}. \tag{2}$$

Since n is odd, we have

$$\frac{F_{n+1}}{F_n} \leq \frac{F_{n+k}}{F_{n+k-1}} \leq \frac{F_{n+2}}{F_{n+1}},$$

so that

$$F_{n+1}F_{n+k-1} \leq F_nF_{n+k} \tag{3}$$

and

$$F_{n+k}F_{n+1} \leq F_{n+k-1}F_{n+2}. \tag{4}$$

Putting Equations (1) and (3) together, we get

$$F_{n+1}^{k} \leq F_{n+k-1}F_{n+1}F_{n}^{k-2} \leq F_{n+k}F_{n}^{k-1}$$

and putting Equations (2) and (4) together, we get

$$F_{n+k}F_{n+1}^{k} \leq F_{n+k-1}F_{n+2}F_{n+1}^{k-1} \leq F_{n+2}^{k}F_{n}$$

as needed.

When n is even, the same argument applies, with all inequalities reversed. $\square$

Proposition 1: *Let k be any positive integer. Then at least the first n terms in the continued fraction expansion of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$ are 1's.*

Proof: Suppose n is odd. By Lemma 3,

$$F_{n+1}^{k}F_n \leq F_{n+k}F_{n}^{k}$$

and

$$F_{n+1}^{k}F_{n+k} \leq F_{n+2}^{k}F_{n}.$$

Therefore,

$$\frac{F_{n+1}^{k}}{F_{n}^{k}} \leq \frac{F_{n+k}}{F_n} \leq \frac{F_{n+2}^{k}}{F_{n+1}^{k}}.$$

Taking k^{th} roots, we get

$$\frac{F_{n+1}}{F_n} \leq \sqrt[k]{\frac{F_{n+k}}{F_n}} \leq \frac{F_{n+2}}{F_{n+1}}.$$

Since $\frac{F_{n+1}}{F_n} = [1, 1, \cdots, 1]$ with n 1's, and $\frac{F_{n+2}}{F_{n+1}} = [1, 1, \cdots, 1, 2]$ with $(n-1)$ 1's, the result follows from Lemma 2.

When n is even, the inequalities are reversed, and the result again follows from Lemma 2. $\square$

3. PROOF OF THEOREM 1

We begin with the case when n is odd. Note that in this case, the truncation of the continued fraction that we are interested in has an *even* number of terms. By Lemmas 1 and 2, it suffices to show that

$$\frac{F_{n+3}(F_{2n+5}+2)+F_{n+2}}{F_{n+2}(F_{2n+5}+2)+F_{n+1}} > \sqrt[5]{\frac{F_{n+5}}{F_n}} > \frac{F_{n+3}(F_{2n+5}+3)+F_{n+2}}{F_{n+2}(F_{2n+5}+3)+F_{n+1}}. \tag{5}$$

By raising all terms to the fifth power, cross-multiplying, and subtracting we reduce the proof of the theorem to showing the following two inequalities.

$$(F_{n+3}(F_{2n+5}+2)+F_{n+2})^5 F_n - (F_{n+2}(F_{2n+5}+2)+F_{n+1})^5 F_{n+5} > 0, \tag{6}$$

$$F_{n+5}(F_{n+2}(F_{2n+5}+3)+F_{n+1})^5 - F_n(F_{n+3}(F_{2n+5}+3)+F_{n+2})^5 > 0. \tag{7}$$

Since $\overline{\phi} = -\phi^{-1}$, Binet's formula becomes

$$F_m = \begin{cases} \frac{\phi^m+\phi^{-m}}{\sqrt{5}} & \text{when } m \text{ is odd} \\ \frac{\phi^m-\phi^{-m}}{\sqrt{5}} & \text{when } m \text{ is even.} \end{cases} \tag{8}$$

We will first prove (6) for all odd n. Let $x = \phi^n$. Using Equation (8), we can rewrite the expression on the left in terms of ϕ and x. After multiplying through by $\sqrt{5}^{11}$, we obtain

$$\begin{aligned} &\Big[(x\phi^3 - x^{-1}\phi^{-3})((x^2\phi^5 + x^{-2}\phi^{-5}) + 2\sqrt{5}) + \sqrt{5}(x\phi^2 + x^{-1}\phi^{-2})\Big]^5 \\ &\qquad \cdot (x + x^{-1}) \\ &- \Big[(x\phi^2 + x^{-1}\phi^{-2})((x^2\phi^5 + x^{-2}\phi^{-5}) + 2\sqrt{5}) + \sqrt{5}(x\phi - x^{-1}\phi^{-1})\Big]^5 \\ &\qquad \cdot (x\phi^5 - x^{-1}\phi^{-5}). \end{aligned} \tag{9}$$

Now, we rewrite ϕ^{-1} as $(\phi - 1)$ and $\sqrt{5}$ as $2\phi - 1$. Our expression becomes a Laurent polynomial in x and ϕ.

Using Mathematica, we evaluate this Laurent polynomial, then impose the relation $\phi^2 = \phi + 1$, and finally set $\phi = \frac{1+\sqrt{5}}{2}$. In order to report our data in a simple way, we round each coefficient to a smaller integer (so that -.1 gets rounded to -1, for example). Thus when x is positive, our expression is bounded below by

$$10746374x^{10} + 15925499x^8 + 6371622x^6 - 528554x^4 - 670244x^2$$

$$-70045 - 2869x^{-2} - 54x^{-4} - 3x^{-6} - x^{-8} - x^{-10}.$$

The absolute value of the sum of the negative coefficients above is 1271771, which is less than the leading coefficient. Since $x = \phi^n > 1$, it follows that this expression is positive for all positive odd values of n, as needed.

In a similar way, we can prove (7) for all odd n. Using the same techniques as described above, we obtain the following Laurent polynomial:

$$10223750003x^{10} + 36597500x^8 + 48841902x^6 + 29462008x^4$$
$$+7371545x^2 + 427815 + 15920x^{-2} + 758x^{-4} + 27x^{-6}.$$

Here, all the coefficients are positive, so the expression is positive for all $x > 0$.

The case when n is even is similar. By Lemmas 2 and 1, it suffices to show that

$$\frac{F_{n+3}(F_{2n+5}-4)+F_{n+1}}{F_{n+2}(F_{2n+5}-4)+F_n} > \sqrt[5]{\frac{F_{n+5}}{F_n}} > \frac{F_{n+3}(F_{2n+5}-3)+F_{n+1}}{F_{n+2}(F_{2n+5}-3)+F_n}. \tag{10}$$

As before, we reduce this to two inequalities.

$$\begin{aligned} &(F_{n+3}(F_{2n+5}-4)+F_{n+1})^5 F_n - (F_{n+2}(F_{2n+5}-4)+F_n)^5 F_{n+5} > 0, \\ &F_{n+5}(F_{n+2}(F_{2n+5}-3)+F_n)^5 - F_n(F_{n+3}(F_{2n+5}-3)+F_{n+1})^5 > 0. \end{aligned} \tag{11}$$

Using the techniques above, we obtain the following two Laurent polynomials:

$$10223750x^{10} - 36597501x^8 + 48841902x^6 - 29462009x^4$$
$$+7371545x^2 - 427816 + 15920x^{-2} - 759x^{-4} + 27x^{-6} - x^{-8},$$

$$10746374x^{10} - 15925500x^8 + 6371622x^6 + 528553x^4$$
$$-670244x^2 + 70044 - 2869x^{-2} + 53x^{-4} - 3x^{-6} - x^{-10}.$$

In the first case, since $x = \phi^n$, and $n \geq 2$, it follows that $x^2 \geq \phi^4 > 6.8$. So, it suffices that the leading coefficient, multiplied by 6.8, be greater than the absolute value of the sum of the negative coefficients, which is 66488086. This is easily seen to be the case. The second case follows similarly.

4. LARGER VALUES OF k

Theorem 1 naturally leads one to wonder about larger values of k. In fact, we have observed interesting behavior for values of k such as 13, 34, 89, and 233, or alternating Fibonacci numbers. In all of these cases, a large number follows a sequence of 1's, possibly with a 2 close to the end. But unlike when $k = 5$, the large number remains constant as n grows, at least for n sufficiently large.

We believe the techniques of this paper could be used to verify each of the observations below, but in hopes that the reader may be inspired to elucidate a deeper structure lurking beneath, we simply present them as empirical data.

Observation 1: *When* $n \geq 15$*, the continued fraction expansion of* $\sqrt[13]{\frac{F_{n+13}}{F_n}}$ *begins with* $n+2$ *1's, then a 2, then a 1, then 377.*

Observation 2: *When* $n \geq 19$*, the continued fraction expansion of* $\sqrt[34]{\frac{F_{n+34}}{F_n}}$ *begins with* $n+4$ *1's, then 12921.*

Observation 3: *When* $n \geq 21$*, the continued fraction expansion of* $\sqrt[89]{\frac{F_{n+89}}{F_n}}$ *begins with* $n+4$ *1's, then a 2, then a 1, then 17710.*

Observation 4: *When* $n \geq 27$*, the continued fraction expansion of* $\sqrt[233]{\frac{F_{n+233}}{F_n}}$ *begins with* $n+6$ *1's, then 606966.*

We note that 377 is the 14th Fibonacci number, and 17711 is the 22nd Fibonacci number. We suspect that the continued fraction expansions of $(\frac{F_{n+k}}{F_n})^{1/k}$ have similar behavior when $k = F_m$ for all large odd integers m, but we have been unable to find a general pattern with which to formulate a conjecture.

ACKNOWLEDGMENT

Theorem 1 was originally conjectured at Williams College in 1993 by members of the Number Theory Group [2] of the SMALL Undergraduate Research Project. That group, under the direction of Edward Burger, studied relations between the Fibonacci sequence and continued fractions. Some of the results we found that summer were later published in [1].

REFERENCES

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