LINEAR EQUALITIES IN FIBONACCI NUMBERS

Presented on the Occasion of Harold Stark's 65th Birthday

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ABSTRACT

We study the equation $F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(m)}$ with $m \geq 3$, $0 < x(1) < b$, and $0 < x(3) < x(4) < \cdots < x(m)$. This equation naturally arises in the generalization of several problems that have appeared in The Fibonacci Quarterly in the problem sections. This equation also has intrinsic interest in its own right. The main theorem - the Accident theorem–states, that under very mild conditions, solutions to this equation cannot happen by accident; that is, there are no singular solutions, but rather every solution belongs to a parametrizable class of solutions. Furthermore if $m \geq 4$, then b must be even and there are exactly 9 parametrizable solutions. This is the first major theorem in the literature on identities in Fibonacci numbers with an arbitrary number of summands whose subscripts have mixed signs. There are only 2 hypothesii of the accident theorem: We require that for all i, $x(i) > 2$ and that no proper subset of summands on the right hand side of the equation has a sum of zero. While the proof of the accident theorem requires many sub-theorems and lemmas, the basic proof method is to exploit *Fibonacci Telescoping* lemmas. An example of *Fibonacci Telescoping* is illustrated by the following identity which leads to one of the 9 parametrizable solutions for $m \geq 4 : F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + \cdots + F_{-(b-1)}$ with b even and o an arbitrary positive odd integer.

1. INTRODUCTION AND SUMMARY

Arising as the generalization of several problems that have appeared in the Fibonacci Quarterly, the equation

$$\left.\begin{array}{l} F_n = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(m)}, \\ 0 < x(1) < b, \quad 0 < x(3) < x(4) < \cdots < x(m), m \geq 3, \end{array}\right\} \tag{1.1}$$

leads to the first major theorem in the literature on identities in Fibonacci numbers with an arbitrary number of summands whose subscripts have mixed signs.

The first problem related to (1.1) was problem B-799 [1]: Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 6F_{n-b} + F_{n-c} + F_{n-d}. \tag{1.2}$$

The problem editor, when publishing the solution [3], stated that most solvers pulled the answer out of a hat, with methods that do not seem to generalize. He challenged the readership with B-804, which replaced 6 in (1.2) with 9342.

$$F_n = F_{n-a} + 9342F_{n-b} + F_{n-c} + F_{n-d} \tag{1.3}$$

The solution to B-804 [2] used identity (1.4) to write (1.5):

$$F_n = F_{n-u+v} + (L_u - L_v)F_{n-u} + F_{n-u-v} + F_{n-2u} \tag{1.4}$$

$$F_n = F_{n-15} + 9342\ F_{n-19} + F_{n-23} + F_{n-38}. \tag{1.5}$$

The above problems naturally suggest the following generalizations: Fix an integer $m \geq 3$. Describe all integer $(m+1)$-tuples— $\{c \neq 0, a(1), \ldots, a(m)\}$ with $0 < a(1) < a(2) < \cdots < a(m)$ such that for all integer $n > 0$,

$$F_n = F_{n-a(1)} + cF_{n-a(2)} + F_{n-a(3)} + F_{n-a(4)} + \cdots + F_{n-a(m)}. \tag{1.6}$$

We call m the *size* of (1.6).

We first *reduce* (1.6) by letting $n = a(2)$. More specifically define

$$b = a(2), x(1) = b - a(1), x(i) = a(i) - b,\ i = 3, 4, \ldots, m. \tag{1.7}$$

The equation (1.1) is the *reduced* equation arising from (1.6). The major task left is to solve the reduced equation (1.1).

The following definitions will be used throughout the paper. A solution of (1.1) is called *large* if b and all $x(i)$ are strictly larger than 2. Similarly, a Fibonacci number, F_z, is called *large* if $|z| > 2$. By abuse of notation we will call a subscript *large* if its absolute value is greater than 2. A solution of (1.1) is called *prime* if there is no non-trivial proper sub-identity (equivalently, if no sum of summands on the right hand side equals zero). More generally, an arbitrary identity is called *prime* if there is no non-trivial proper sub-identity. A solution of (1.1) is called *1-parametrizable* if increasing all subscripts by an arbitrary even number also yields a solution.

Note, that if a solution of (1.1) is *1-parametrizable* then the infinite class of identities to which it belongs can be parametrized with parameter b and constants $a(i)$ as defined by (1.7). If a solution to (1.1) is not *1-parametrizable* then it is called *singular*. A solution of (1.1) is called *even* (or *odd*) according to the parity of b. In section 2 we solve (1.1) for size 3, illustrate examples of these concepts and motivate their relevance to the main theorem.

In section 3 we present the main theorem of this paper, The Accident Theorem, which states that for each $m \geq 3$, all large prime solutions of (1.1) are 1-*parametrizable*; furthermore, if $m \geq 4$ then all *large prime* solutions occur in one of nine forms and b must be even. Theorem 3.1 is named "the accident theorem" since it shows that a solution to (1.1) cannot happen by accident but has to be part of a larger class of identities. Section 4 contains three lemmas useful in the proof of Theorem 3.1 as presented in sections 2 and 5-8.

2. SIZE 3

When $m = 3$, (1.1) reduces to $F_b = F_{x(1)} + F_{-x(3)},\ 0 < x(1) < b;\ 0 < x(3)$. We have

Lemma 2.1: If $\{b, x(1), x(2)\}$ is a *large* solution to (1.1) for $m = 3$, then either

$$x(1) = b - 1, x(2) = b - 2, b \text{ odd}, b \geq 5$$

or

$$x(1) = b - 2, x(2) = b - 1, b \text{ even}, b \geq 5.$$

Proof: We divide the proof into cases according to the value of $x(1)$. Note, that by (1.1) $x(1) < b$.

Case $x(1) = b-1$ **or** $x(1) = b-2$: If $x(1) = b-1$ then by (1.1) $F_{-x(3)} = F_b - F_{x(1)} = F_{b-2}$, and therefore by (1.1) we must have that $x(3)$ is odd with $x(3) = b-2$. The requirement $b \geq 5$ assures that the solution is *large*. The proof for the case $x(1) = b-2$ is almost identical.

Case $x(1) \leq b-3$: But then either $F_{-x(3)} \leq F_{b-1}$, and therefore $F_{x(1)} + F_{-x(3)} \leq F_{b-3} + F_{b-1} < F_b$, a contradiction for $b, x(i) > 2$, or else, $F_{-x(3)} \geq F_b$ and since $x(1)$ is positive, $F_{x(1)} + F_{-x(3)} > F_b$, a contradiction.

Theorem 2.2: The only solutions to (1.1) for *size* $m = 3$ are

$$\left.\begin{array}{ll} x(1) = b-1,\ x(2) = b-2, & b \geq 3 \text{ and } b \text{ odd}, \\ x(1) = b-2,\ x(2) = b-1, & b \geq 4 \text{ and } b \text{ even}, \end{array}\right\} \tag{2.1}$$

$$\left.\begin{array}{lll} b = 3, & x(1) = 1, & x(2) = -1 \\ b = 4, & x(1) = 1, & x(2) = -3 \\ b = 4, & x(1) = 3, & x(2) = -1. \end{array}\right\} \tag{2.2}$$

Proof: If a solution is not *large* then either b or some $x(i)$ equals 1 or 2. It therefore suffices to computationally check (1.1) for all integer-3-tuples in the $[1,4]^3$ cube in R^3. A computer check of these 64 points reveals the 3 solutions mentioned in (2.2) and also the *non-large* solutions $F_3 = F_2 + F_{-1}$ and $F_4 = F_2 + F_{-3}$.

The main focus of this paper is on solving (1.1). Corollary 2.3 shows how solutions of (1.1) yield solutions to (1.6) using the *reduction* method defined by (1.7). We do not further pursue the study of (1.6) in this paper.

Corollary 2.3: All solutions to (1.6) for $m = 3$ are in one of the following forms:

$$\begin{array}{ll} F_n = F_{n-1} + L_{b-2}F_{n-b} + F_{n-(2b-2)}, & b \geq 3 \text{ and } b \text{ odd} \\ F_n = F_{n-2} + L_{b-1}F_{n-b} + F_{n-(2b-1)}, & b \geq 4 \text{ and } b \text{ even} \\ F_n = F_{n-1} + 2F_{n-4} + F_{n-5} & \\ F_n = F_{n-2} + 2F_{n-3} + F_{n-4} & \\ F_n = F_{n-3} + 5F_{n-4} + F_{n-7} & \end{array}$$

Note that the *non-large* solutions in (2.2) are *singular* while the *large* solutions in (2.1) are *1-parametrizable*. Similarly the following solution of (1.1), for size 4, $F_b = F_{x(1)} + F_{-x(1)} + F_{-b}$, $0 < x(1) < b$ b odd, $x(1)$ even, is neither *prime* nor *singular* and can heuristically be thought of as *factoring* into the two identities: $F_b = F_{-b}$ and $F_{x(1)} + F_{-x(1)} = 0$. In a similar manner the solution $F_b = F_1 + F_{-2} + F_{-b}$, b odd, shows how *non-large non-prime* solutions may be *singular*. These and similar considerations motivate only considering *large prime* solutions to (1.1) in the main theorem.

3. THE ACCIDENT THEOREM

We are now in a position to state the main theorem. First we state some

Notational Conventions: For the rest of the paper the symbols o, o', o'' will stand for arbitrary, odd, positive integers. If J is a set of integers, then $a \leq J \leq b$ means that $a \leq j \leq b$ for all $j \in J$. Similarly the statement that J is odd or even means all members of J are odd or even respectively. For the rest of the paper equality of sets by convention means equality of ordered sets. If the string $F_x + F_y + \cdots + F_z$ occurs in an equation then this string either refers to (a) the single summand F_x or (b) the sum $F_x + F_z$ or (c) the sum $F_x + F_{x+d} + F_{x+2d} + \cdots + F_{x+jd}$ with $y = x+d, d \neq 0$ and $z = x+jd$ for some positive non-zero integer $j > 1$ (d is allowed to be negative). In particular, by convention, throughout this paper, such a string is assumed non-empty. A string of the form $F_x + F_y + \cdots + F_z + F_u$ is interpreted as $(F_x + F_y + \cdots + F_z) + F_u$; similarly a string of the form $F_x + F_y + \cdots + F_z + F_u + F_v + \cdots + F_w$ is interpreted as $(F_x + F_y + \cdots + F_z) + (F_u + F_v + \cdots + F_w)$. These conventions allow an unambiguous interpretation of every string with possibly multiple triple dot notations. Similar conventions apply to sequences.

The Accident Theorem 3.1: Suppose, for some $m \geq 3$, that $\{b, x(1), x(3), x(4), \ldots, x(m)\}$ is a *large prime* solution of (1.1). Then this solution is *1-parametrizable*. Furthermore either $m = 3$ and (2.1) holds or else $m > 3$, b is even, and this solution is in one of the following 9 forms:

Form 1: $F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + \cdots + F_{-(b-1)}$

Form 2: $F_b = F_{b-2} + F_{-b} + F_{-(b+1)}$

Form 3: $F_b = F_{b-2} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + \cdots + F_{-(b+o''+1)} + F_{-(b+o''+2)}$

Form 4: $F_b = F_{b-o-3} + F_{-(b-o-1)} + F_{-(b-o)} + \cdots + F_{-(b-1)} + F_{-b} + F_{-(b+1)}$

Form 5: $F_b = F_{b-o-3} + F_{-(b-o-1)} + F_{-(b-o)} + \cdots + F_{-(b-1)} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + \cdots + F_{-(b+o''+1)} + F_{-(b+o''+2)}$,

Form 6: $F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + \cdots + F_{-(b-3)} + F_{-b} + F_{-(b+1)}$

Form 7: $F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + \cdots + F_{-(b-3)} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + \cdots + F_{-(b+o''+1)} + F_{-(b+o''+2)}$,

Form 8: $F_b = F_{b-o-4-o'} + F_{-(b-o-3-o')} + F_{-(b-o-1-o')} + \cdots + F_{-(b-o-4)} + F_{-(b-o-1)} + F_{-(b-o)} + \cdots + F_{-(b-1)} + F_{-b} + F_{-(b+1)}$

Form 9: $F_b = F_{b-o-4-o'} + F_{-(b-o-3-o')} + F_{-(b-o-1-o')} + \cdots + F_{-(b-o-4)} + F_{-(b-o-1)} + F_{-(b-o)} + \cdots + F_{-(b-1)} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + \cdots + F_{-(b+o''+1)} + F_{-(b+o''+2)}$.

Conversely every choice of positive, *large*, even integer b and every choice of o, o' and o'' for which all subscripts are *large*, yield, for some $m > 3$, a *1-parametrizable, large, even, prime* solution to (1.1).

Examples: The following solutions to (1.1), for various sizes of m, illustrate the 9 forms.

Form 1: $F_b = F_{b-10} + F_{-(b-9)} + F_{-(b-7)} + F_{-(b-5)} + F_{-(b-3)} + F_{-(b-1)}$

Form 2: $F_b = F_{b-2} + F_{-b} + F_{-(b+1)}$

Form 3: $F_b = F_{b-2} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + F_{-(b+6)} + F_{-(b+7)}$

Form 4: $F_b = F_{b-4} + F_{-(b-2)} + F_{-(b-1)} + F_{-b} + F_{-(b+1)}$

Form 5: $F_b = F_{b-4} + F_{-(b-2)} + F_{-(b-1)} + F_{-b} + F_{-(b+2)} + F_{-(b+3)}$

Form 6: $F_b = F_{b-8} + F_{-(b-7)} + F_{-(b-5)} + F_{-(b-3)} + F_{-b} + F_{-(b+1)}$

Form 7: $F_b = F_{b-6} + F_{-(b-5)} + F_{-(b-3)} + F_{-b} + F_{-(b+2)} + F_{-(b+3)}$

Form 8: $F_b = F_{b-6} + F_{-(b-5)} + F_{-(b-2)} + F_{-(b-1)} + F_{-b} + F_{-(b+1)}$

Form 9: $F_b = F_{b-8} + F_{-(b-7)} + F_{-(b-5)} + F_{-(b-2)} + F_{-(b-1)} + F_{-b} + F_{-(b+2)} + F_{-(b+3)}$.

Certain patterns emerge in the nine forms that will facilitate proving theorem 3.1.

- Only form 1 has $x(i) < b$, all i (the all-below-b case).
- In all other forms $x(i) \geq b$ for some i.
- The forms naturally occur in pairs: The set of $x(i)$ above b, $\{x(i) : x(i) > b\}$, equals either $\{b+1\}$, or $\{b+2, b+4, \ldots, b+o''+1, b+o''+2\}$
- In a similar manner the set of $x(i)$ below b, $\{x(i) : x(i) < b\}$ equals one of four forms.

Using these observations we now outline the proof of theorem 3.1:

- The case $m = 3$ is covered in Lemma 2.1.
- (Corollary 4.2)The nine forms, for any even b, and any choice of positive o, o', o'', are solutions of (1.1).
- A solution of (1.1) either has $x(i) < b$ for all i or else there exists some i such that $x(i) \geq b$.
- The All below b case: (Theorem 5.6) If all $x(i) < b$ and $m > 3$ then form 1 holds.
- The Above b case: (Theorem 6.2, Corollary 6.3) If there exists some j_0 such that $x(j_0) = b$ then $\{x(i) : i > j_0\}$ equals either
  - $\{x(m)\} = \{b+1\}$, or
  - $\{x(j_0+1),\ x(j_0+2), \ldots, x(m-1), x(m)\} = \{b+2, b+4, \ldots, b+o''+1, b+o''+2\}$.
- (Corollary 6.3(c)) If there exists some i such that $x(i) \geq b$ then in fact there is some j_0 such that $x(j_0) = b$
- The Below b case: (Theorem 7.4) If there exists some j_0 such that $x(j_0) = b$, then $\{x(i) : i < j_0\}$, equals either
  - $\{x(1)\} = \{b-2\}$ or
  - $\{x(1), x(3), x(4), \ldots, x(j_0-1)\} = \{b-o-3, b-o-1, b-o, \ldots, b-1\}$ or
  - $\{x(1), x(3), x(4), \ldots, x(j_0-1)\} = \{b-o-3, b-o-2, b-o, \ldots, b-3\}$ or
  - $\{x(1), x(3), x(4), \ldots, x(j_0-1)\} = \{b-o-4-o', b-o-3-o', b-o-1-o', \ldots, b-o-4, b-o-1, b-o, \ldots, b-1\}$
- The odd-b case: (Theorem 8.7) For $m > 3$, there is no *large prime odd* solution of (1.1).

4. THREE USEFUL LEMMAS

The proof of the accident theorem is greatly simplified by the following three lemmas. The names given these lemmas will be used in the sequel.

Lemma 4.1 (Fibonacci Telescoping): For any integer z we have

$$F_z + F_{z+1} + F_{z+3} + \cdots + F_{z+o} = F_{z+o+1}$$

$$F_z - F_{z-1} - F_{z-3} - \cdots - F_{z-o} = F_{z-o-1}.$$

Corollary 4.2: Let b be any even number. Let o, o' and o'' be arbitrary positive odd integers. Then the 9 forms listed in section 3, yield solutions to (1.1).

Proof: We prove that form 9 yields a solution, the proof for the other forms being similar. By the evenness of b and Lemma 4.1 we have

$$F_{(b+1)} = F_{-(b+2)} + F_{-(b+4)} + \cdots + F_{-(b+o''+1)} + F_{-(b+o''+2)},$$

and

$$F_{(b-o-3)} = F_{b-o-4-o'} + F_{-(b-o-3-o')} + F_{-(b-o-1-o')} + \cdots + F_{-(b-o-4)}.$$

Clearly $F_{(b-o-3)} + [F_{-(b-o-1)} + F_{-(b-o)}] + \cdots + F_{-(b-1)} = F_{(b-o-3)} + F_{(b-o-2)} + F_{(b-o)} + \cdots + F_{(b-3)}$ which equals $F_{(b-2)}$ by Lemma 4.1. Hence the right hand side of form 9 reduces to $F_{(b-2)} + F_{-b} + F_{(b+1)} = F_b$ as was to be shown.

Lemma 4.3 (Parity Transposition):

(a) Let J be a set of positive integers of the same parity with $2 < J \leq z$. Then $\sum_{j \in J} F_j < F_{z+1}$

(b) Suppose further that for some k, of the same parity as members of J, with $2 < k \leq z$ that $k \notin J$. Then $\sum_{j \in J} F_j < F_{z+1} - F_k$.

Proof of part (a): Since $j > 2$, the lemma trivially follows from the following well known identities (e.g. [4]): $F_1 + F_3 + F_5 + \cdots + F_o = F_{o+1}, F_2 + F_4 + F_6 + \cdots + F_{o-1} = F_o a - 1$. Part (b) trivially follows from part (a).

Lemma 4.4: (Alternating Fibonacci Telescoping) If all summands are *large* then $F_o - F_{o-1} + F_{o-2} - F_{o-3} \cdots < F_{o-1}$.

Proof: A straightforward induction shows that $-F_2 + F_3 - F_4 + \cdots + F_o = F_{o-1},\ o$ odd with strict equality. Consequently an alternating sum that avoids small subscripts must have strict inequality.

5. THE ALL-BELOW-b CASE

Throughout this section we assume that for some integers $n, z, u(1), u(3), \ldots, u(n)$, that

$$\left.\begin{array}{l} F_z = F_{u(1)} + F_{-u(3)} + F_{-u(4)} + \cdots + F_{-u(n)} \text{ is a \textit{large prime even} identity.} \\ \text{with } u(1) < z,\ u(3) < u(4) < \cdots < u(n) < z, \quad n \geq 3 \end{array}\right\} \tag{5.1}$$

Lemma 5.1:

(a) There exists some j such that $u(j) = z - 1$

(b) It is not possible for both $u(1) = z - 1$ and $u(n) < z - 1$.

Proof of (a): Assume to the contrary that $u(i) \leq z - 2$ for all i. Then the odd $u(i)$ are bounded above by $z - 3$. After transposing negative summands from the right side of (5.1) to the left side we have an equation of the form $F_z + \sum_{i \in K \leq z-2} F_i = F_{u(1)} + \sum_{i \in J \leq z-3} F_i$ for some sets of integers, J, K, of odd and even parity respectively. Applying Lemma 4.3 we obtain $F_z \leq F_z + \sum_{i \in K \leq z-2} F_i = F_{u(1)} + \sum_{i \in J \leq z-3} F_i \leq F_{z-2} + F_{z-2}$ which is false for even $z > 2$. This contradiction shows our original assumption false and completes the proof of Lemma 5.1(a).

The proof of Lemma 5.1(b) is almost identical to the proof of Lemma 5.1(a) and hence omitted.

Remark: The above argument could have, without serious loss of detail, been succinctly stated as follows: 'Proof of part (a): If we assume to the contrary that $u(i) \leq z - 2$, then a *parity transposition* shows $F_z + \sum_{i \in K} F_i = F_{u(1)} + \sum_{i \in J \leq z-3} F_i < F_{z-2} + F_{z-2}(*)$.' The definitions and upper bounds for the sets J and K are clear from context. In the sequel we will use similar shorter forms for proofs.

Lemma 5.2:

(a) If for some $p \geq 3$, $0 = F_{-u(3)} + \cdots + F_{-u(p)} + F_{z-o},\ u(i) \leq z - o + 1$, then $u(p) = z - o + 1$

(b) If for some $p \geq 3$, $F_{z-(o+1)} = F_{-u(3)} + \cdots + F_{-u(p)}$, $u(i) \leq z - (o+1) + 1$ then $u(p) = z - (o+1) + 1$.

Proof of (a): Assuming to the contrary that the even $u(i)$ are bounded by $z - o - 1$ and applying a parity transposition to the equation in the statement of Lemma 5.2(a) we infer $F_{z-o} \leq F_{z-o} + \sum_{i \in J} F_i = \sum_{i \in K \leq z-o-1} F_i < F_{z-o}(*)$. This contradiction shows that the largest $u(i)$ is in fact $z - o + 1$. The proof is completed by noting that by (5.1) the largest $u(i)$ occurs at $i = p$. (The proof of part (b) is almost identical and hence omitted).

Lemma 5.3: We cannot have both $u(1) = z - 1$ and $u(n) = z - 1$.

Proof: Assume to the contrary. Substituting, $u(1) = z - 1$ and $u(n) = z - 1$ into (5.1) yields

$$0 = F_{-u(3)} + F_{-u(4)} + \cdots + F_{-u(n-1)} + F_{z-3} \tag{5.2}$$

For $3 \leq i \leq n-1$, we have $u(i) \leq z - 2$, since $u(i) \leq u(n-1) < u(n) = z - 1$. Applying Lemma 5.2(a) to (5.2) shows that $u(n-1) = z - 2$ and therefore by the evenness of z, $F_{-(z-3)} + F_{-u(n-1)} = F_{z-3} - F_{z-2} = -F_{z-4}$. Hence simplifying (5.2) yields

$$F_{z-4} = F_{-u(3)} + \cdots + F_{-u(n-2)}, \ u(i) \leq z - 3.$$

If $n - 2 \geq 3$ then we may apply Lemma 5.2(b) showing $u(n-2) = z - 3$.

Subtracting F_{z-4} from both sides of this last equation yields $0 = F_{-u(3)} + \cdots + F_{-u(n-3)} + F_{z-5}$, $u(i) \leq z - 4$. If $n - 3 \geq 3$ then we may again apply Lemma 5.2(a) showing that $u(n-3) = z - 4$. We also have, $F_{z-5} + F_{-(z-4)} = -F_{z-6}$. If $n - 4 \geq 3$ we may transpose F_{z-6} to the left side and again apply Lemma 5.2(b) showing that $u(n-4) = z - 5$.

This process may be continued inductively yielding $u(j) = z - (n+1-j)$, for $j = 3, 4, \ldots, n$. Substituting these values back into (5.2) and using the assumption $u(1) = z - 1$ we have $F_z = F_{z-1} + (F_{z-1} - F_{z-2} + F_{z-3} \ldots)$. But by the Alternating Fibonacci Telescoping, Lemma 4.4, the right side of this last equation is strictly bounded above by $F_{z-1} + F_{z-2} = F_z$ a contradiction. This contradiction shows that our original assumption that both $u(1)$ and $u(n)$ equal $z - 1$ is false and completes the proof.

For future reference we note that we have proven

Corollary 5.4: Equation (5.1) implys (5.2) is not possible.

Lemma 5.5: If (5.1) holds then

(a) $u(n) = z - 1$

(b) If $n > 3$ then $u(1) < z - 2$.

(c) If $n > 3$ then $u(i) \leq z - 3$ for $i = 3, 4, \ldots, n - 1$.

Proof of (a): By Lemma 5.1(a), $u(j) = z - 1$ for some j. By (5.1) either $j = 1$ or $j = n$. By Lemmas 5.1(b) and 5.3, $j \neq 1$. Hence $j = n$.

Proof of (b): First, by Lemma 5.5(a) and Lemma 5.3, $u(1) \neq z - 1$. Next, if contrary to Lemma 5.5(b), $u(1) = z - 2$ then by Lemma 5.5(a) and (5.1), $F_z = F_{u(1)} + F_{-u(n)} = F_{z-2} + F_{z-1}$ is a strictly proper sub-equation of (5.1) violating primality, since $n > 3$. Hence our original assumption is false and $u(1) < z - 2$ as required.

Proof of (c): Assume to the contrary that $u(n-1) = z - 2$. By Lemmas 5.5(a), (b), $u(n) = z - 1$ and $u(1) \leq z - 3$ respectively. Applying a parity transposition to (5.1) yields

$$F_z + \sum_{i \in J} F_i = F_{u(1)} + \sum_{i \in K \leq z-3} F_i + F_{u(n-1)} + F_{u(n)} < F_{z-3} + F_{z-1} = F_z(*).$$

We conclude for $3 \leq i \leq n-1$ that $u(i) \leq u(n-1) < z-2$ as required.

We apply Lemmas 5.1 - 5.5, to deal with the <u>all-below-b case</u> of the main accident theorem.

Theorem 5.6 (the <u>all below-b-case</u>): Suppose for some $m \geq 3$ that $\{b, x(1), x(3), x(4), \ldots, x(m)\}$ is a prime, large, even solution to (1.1). Suppose further that $x(i) < b$, all i. Then (for some positive odd integer o), $\{x(1), x(3), x(4), \ldots, x(m)\} = \{b-o-1, b-o, b-o+2, \ldots, b-1\}$

Proof: Applying Lemma 5.5(a) with $n = m,\ z = b,\ u(i) = x(i)$ shows $x(m) = z-1$. If $m = 3$ then $F_{x(1)} = F_b - F_{-x(3)} = F_b - F_{b-1} = F_{b-2}$ and we are done. If $m > 3$ then subtracting $F_{z-1} = F_{b-1}$ from both sizes of (1.1) yields

$$F_{b-2} = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(m-1)}.$$

By Lemma 5.5(b)-(c), $x(i) < b-2$ for all i. Therefore since $m-1 \geq 3$ then we may again apply Lemma 5.5(a) with $n = m-1,\ z = b-2,\ u(i) = x(i)$, showing that $x(m-1) = z-1 = b-3$. If $m-2 \geq 3$ we may subtract F_{b-3} from both sides, apply again Lemma 5.5(b)-(c) to show that $x(i) < b-4$ and then apply again Lemma 5.5(a), with $z = b-4,\ u(i) = x(i),\ n = m-2$, showing that $x(m-2) = b-5$. We may continue this process inductively showing that

$$F_b = F_{x(1)} + F_{-(b-o)} + F_{-(b-o+2)} + \cdots + F_{-(b-3)} + F_{-(b-1)}.$$

Hence by Fibonacci Telescoping, $x(1) = b-o-1$ completing the proof of Theorem 5.6.

6. THE ABOVE-b CASE

In this section we show that if in (1.1), $x(i) \geq b$, for some i, then in fact $\{x(i) : x(i) > b\}$ equals either $\{b+1\}$ or $\{b+2, b+4, \ldots, b+o''+1, b+o''+2\}$.

Throughout this and the next section we assume that for some integers $n,\ z,\ u(k),\ k = 1, 3, 4, \ldots,$ that

$$\left.\begin{array}{l} F_z = F_{u(1)} + F_{-u(3)} + F_{-u(4)} + \cdots + F_{-u(n)} \quad \text{is a } \textit{large, prime, even} \text{ identity, with} \\ u(1) < z;\ u(3) < u(4) < \cdots < u(n), n \geq 3, \text{ with } u(i) \geq z \text{ for some } i \geq 3, \end{array}\right\} \tag{6.1}$$

Lemmas 6.1: If (6.1) holds then

(a) For some $j,\ u(j) > z$.

(b) Sup $u(j) - z$ is odd; the supremum occurs at $j = n$

(c) Suppose $u(n) = z + o$. If $o \geq 3$ then $u(n-1) = z + o - 1$.

(d) Suppose $u(n) = z + o,\ o \geq 3$, and $u(n-1) = z + o - 1$. Then $u(n-2) < z + o - 2$.

Proof of (a): If, to the contrary $u(j) \leq z$ for all j, then by (6.1) $u(i) = z$ for some $i \geq 3$ and hence, by (6.1) $u(n) = z$. Applying a parity transposition to (6.1) yields $F_z + F_z + \sum_{i \in K \leq z-2} F_i = F_{u(1)} + \sum_{i \in J \leq z-1} F_i < F_{z-1} + F_z (*)$.

Proof of (d): If to the contrary $u(n) = z+o,\ u(n-1) = z+o-1$, and $u(n-2) = z+o-2$ then applying a parity transposition yields $F_z + F_{z+o} > F_z + F_{z+o-1} + \sum_{i \in J \leq z+o-3} F_i = F_{u(1)} + \sum_{i \in K} F_i + F_{z+o-2} + F_{z+o} \geq F_{z+o} + F_{z+o-2}$, a contradiction for $o \geq 3$.

The proofs of parts (b) and (c) are almost identical to the proofs of parts (a) and (d) and hence omitted.

Theorem 6.2 (The above-b-case): Suppose $b, x(1), x(3), x(4), \ldots, x(m), m \geq 4$, is a *large, prime, even* solution of (1.1). Suppose further that $x(i) \geq b$ for some $i \neq 2$. Then either

$$x(m) = b + 1 \tag{6.2}$$

or else for some positive integer j,

$$x(m) = b + o'' + 2, x(m-1) = b + o'' + 1, x(m-2) = b + o'' - 1, \ldots, x(m-j) = b + 2 \tag{6.3}$$

Proof: Applying Lemma 6.1(b), with $z = b,\ u(j) = x(j),\ n = m$, shows that $x(m) = b+o$. If $o = 1$ then (6.2) is satisfied and we are done. If $o > 1$ then since o is odd, $o \geq 3$. Applying Lemma 6.2(c), with $z = b,\ u(j) = x(j),\ n = m$, shows $x(m-1) = b + o - 1$; similarly applying Lemma 6.2(d) shows that $x(m-2) < b + o - 2$. Since b is even and o is odd we have $F_{-x(m)} + F_{-x(m-1)} = F_{-(b+o)} + F_{-(b+o-1)} = F_{-(b+o-2)}$. Hence (1.1) reduces to

$$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(m-2)} + F_{-(b+o-2)}.$$

If $o = 3$ then $x(m) = b+o = b+3, x(m-1) = b+o-1 = b+2$, and (6.3) is satisfied with $o'' = 1$ and $j = 1$. If $o \geq 5$ then since $0 < x(1) < b$ and $x(3) < x(4) < \cdots < x(m-2) < b+o-2$, we may again apply Lemmas 6.1(c)-(d), with $z = b, u(i) = x(i), i = 1, 3, 4, ..., m-2, u(m-1) = b + o - 2, n = m - 1$, showing that $x(m-2) = b + o - 3$ and $x(m-3) < b + o - 4$.

We may inductively continue this process of combining the last 2 terms on the right side and applying Lemmas 6.2(c)-(d) until $x(m-j) = b + 2$ for some j. We then have, $x(m) = b + o, x(m-1) = b + o - 1, x(m-2) = b + o - 3, \ldots, x(m-j) = b + 2$. Letting $o'' = o - 2$ completes the proof of theorem 6.2.

Using Theorem 6.2 we can immediately derive important properties of the subscripts "below" and "above" b.

Corollary 6.3: If, in (1.1), $x(i) \geq b$ for some i, then for some integer j_0,

(a) $\sum_{k \in J} F_{-x(k)} = F_{b+1}$ with $J = \{j_0 + 1, j_0 + 2, \ldots, m\}$,

(b) If k is not in J then $x(k) \leq b$.

(c) $x(j_0) = b$.

(d) $\{x(1), x(3), x(4), \ldots, x(m)\} = S_1 \cup \{x(j_0)\} \cup S_2$ with $S_1 = \{x(j) : j < j_0\}$ and $S_2 = \{x(j) : j > j_0\}$ with $F_{x(1)} + \sum_{j \in S1 - \{x(1)\}} F_{-x(j)} = F_{b-2}$, and $\sum_{j \in S2} F_{-x(j)} = F_{b+1}$ and $x(j_0) = b$. (Note: The set $S_1 - \{x(1)\}$ may be empty.)

Proof: Part (a) follows by applying Fibonacci telescoping to (6.3). Part (d) follows from parts (a)-(c) since applying part (a) and using the evenness of b, we infer that $F_{x(1)} + \sum_{j \in S1 - \{x(1)\}} F_{-x(j)} = F_b - F_{-b} - \sum_{i \in S2} F_{-x(i)} = 2F_b - F_{b+1} = F_{b-2}$. We have left to prove parts (b) and (c).

Proof of (b): The hypothesii of theorem 6.2 imply either (6.2) or (6.3). If (6.2) holds then by (1.1), $x(k) \leq b$, for $k \leq m-1$ as required. If (6.3) holds then by (1.1), $x(k) \leq b+1$ for $k \leq j_0$. Therefore to prove $x(k) < x(j_0) \leq b$, for $k \leq j_0$, it suffices to assume to the contrary that $x(j_0) = b + 1$ and derive a contradiction. By part (a) we can rewrite (1.1) as

$$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(j0+1)} + F_{-x(j0)} + F_{b+1}. \tag{6.4}$$

Applying a parity transposition to (6.4) yields $F_b + F_{b+1} > F_b + \sum_{k \in K \leq b} F_i = \sum_{k \in L} F_k + F_{b+1} + F_{b+1} > 2F_{b+1}(*)$.

Proof of (c): If we assume to the contrary that $x(j_0) < b$, then applying a parity transposition to (6.4) yields $F_{b+1} = F_b + F_{b-1} > F_b + \sum_{k \in L \leq b-2} F_k = F_{x(1)} + \sum_{k \in L} F_k + F_{b+1} > F_{b+1}(*)$.

7. THE BELOW-b CASE

In this section we completely describe the structure of the set S_1 defined in corollary 6.3. Recall

$$F_{x(1)} + \sum_{j \in S1 - \{x(1)\}} F_{-x(j)} = F_{b-2}. \tag{7.1}$$

Define $k = \inf\{i : S_1 \leq x(i)\}$. By Lemma 6.3(d) if $k = 1$ then $x(1) = b - 2$. Accordingly for the rest of this section we assume $k \geq 3$ (Note: When $k \geq 3, k = j_0 - 1$.)

Lemma 7.1: If $x(i) \leq b - 3$, for $1 \leq i \leq k$ then

$$\{x(1), x(3), x(4), \ldots, x(k)\} = \{b - o - 3, b - o - 2, b - o, \ldots, b - 3\}. \tag{7.2}$$

Proof: We apply theorem 5.6 with b replaced by $b - 2$, and m replaced by k. The hypothesii of theorem 7.1 are satisfied since

- The requirement of theorem 5.6 that $m \geq 3$ is satisfied because we assume $k \geq 3$
- $b - 2$ is even because b is even
- $x(1) \leq b - 3 = (b - 2) - 1$ by assumption
- The inequalities $2 \leq x(i) \leq b - 3$, imply $5 \leq b$ or $b - 2 > 2$ as required
- a violation of primality in 7.1 would imply a corresponding violation of primality in (1.1).

The conclusion of theorem 5.6 (with b replaced by $b - 2$) is (7.2). This completes the proof of Lemma 7.1.

Accordingly for the rest of this section we assume

$$x(i) \geq b - 2 \text{ for some } i, 1 \leq i \leq k. \tag{7.3}$$

Lemma 7.2:

(a) $x(i) = b - 1$ for some i.
(b) It is not possible for $x(1) = b - 1 = x(k)$.
(c) It is not possible for $x(1) = b - 1$ and $x(j) \leq b - 2$, for $3 \leq j \leq k$.
(d) $x(k) = b - 1$.

Proof of (a): If to the contrary, $x(i) \leq b - 2$, for all i then by (7.3) $x(i) = b - 2$ for some i. We cannot have $i = 1$ since the equation $F_{x(1)} = F_{b-2}$ coupled with $k \geq 3$ implies a violation of primality in (7.1). We conclude $x(i) = b - 2$ for some $i \geq 3$ and therefore by (1.1), $i = k$ and $x(j) \leq b - 3$ for $j < k$. Applying a parity transformation to (7.1) yields $F_{b-2} + F_{b-2} + \sum_{i \in J} F_i = F_{x(1)} + \sum_{i \in K \leq b-3} F_i < F_{b-3} + F_{b-2} \leq F_{b-2} + F_{b-2}(*)$.

Proof of (b): If we assume to the contrary, then $F_{b-2} = F_{b-1} + F_{-x(3)} + \cdots + F_{-x(k-1)} + F_{b-1}$. Applying a parity transposition to this equation we obtain $F_{b-1} + F_{b-1} + \sum_{i \in J} F_i = F_{b-2} + \sum_{i \in K \leq b-2} F_i < F_{b-2} + F_{b-1}$, a contradiction.

Proof of (c): If we assume to the contrary that $x(1) = b-1$ and $x(j) \leq b-2$, for $j \geq 3$, then subtracting F_{b-2} from both sides of (7.1) yields $0 = F_{b-3} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(k)}$. The result now follows from corollary 5.4.

Proof of (d): By definition if $i \in S_1, x(i) < b$. By (1.1) the largest $x(i)$ occurs at $i = k$ or $i = 1$. The result now follows from (a)-(c).

We continue studying the possible structure of S_1. By Lemma 7.2(d), there is a largest integer, $p,\ 1 \leq p \leq k-2$ such that

$$\{x(k),\ x(k-1), \ldots, x(k-(p-1)\} = \{b-1, b-2, \ldots, b-p\}. \tag{7.4}$$

Clearly, if $k-(p-1) > 3$ then by the maximality of $p,\ x(k-p) \leq b-p-2$. Similarly, if $k-(p-1) = 3$ then $x(1) \leq b-p-2$ since if we assume to the contrary that $k-(p-1) = 3$ and $x(1) = b-p-2$, then substituting (7.4) into (7.1) yields $F_{b-2} = F_{b-1} - F_{b-2} + F_{b-3} - F_{b-4} \ldots$ which by Lemma 4.5 is not possible. We now complete the study of the structure of S_1 by considering three cases according to the parity of p and the size of $k-(p-1)$.

Lemma 7.3: Suppose (7.4) holds. Then

(a) If p is even and $k-(p-1) = 3$ then

$$\{x(1), x(3), x(4), \ldots, x(k)\} = \{b-o-3, b-o-1, b-o, \ldots, b-1\}. \tag{7.5}$$

(b) If p is even and $k-(p-1) > 3$ then for some positive integer j,

$$\left.\begin{array}{l}\{x(1), x(3), x(4), \ldots, x(j), x(j+1), x(j+2), \ldots, x(k)\} \\ = \{b-o-4-o', b-o-3-o', b-o-1-o', \ldots, b-o-4, b-o-1, b-o, \ldots, b-1\}\end{array}\right\}. \tag{7.6}$$

(c) p cannot be odd.

Proof of (a): Using the assumptions we have

$$\begin{aligned} F_{x(1)} &= F_{b-2} - [F_{-x(k)} + F_{-x(k-1)} + \cdots + F_{-x(k-(p-1))}],\ \text{by (7.1)}, \\ &= F_{b-2} - [(F_{b-1} - F_{b-2}) + (F_{b-3} - F_{b-4}) + \cdots + (F_{b-(p-1)} - F_{b-p})],\ \text{by (7.4)}, \\ &= F_{b-2} - F_{b-3} - F_{b-5} - F_{b-7} - \cdots - F_{b-(p+1)}, \\ &= F_{b-(p+2)},\ \text{by Fibonacci Telescoping.} \end{aligned}$$

Letting $o = p+1$ yields (7.5) .

Proof of (b): Proceeding as in the proof of part (a) we have

$$F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(k-p)} = F_{b-(p+2)}. \tag{7.7}$$

First we show that $x(j) \leq b-(p+3)$ for $3 \leq j \leq k-p$. By the maximality of p in (7.4) we have $x(j) \leq b-(p+2)$ for $3 \leq j \leq k-p$. Furthermore, if $x(k-p) = b-(p+2)$ exactly then (7.7) would imply a violation of primality in (7.1) and hence a corresponding violation of primality in (1.1). We conclude $x(j) \leq b-(p+3)$.

Next we show that $x(1) \leq b-(p+3)$. On the one hand if $x(1) = b-(p+2)$ then (7.7) implies a violation of primality; on the other hand if $x(1) \geq b-(p+1)$, then a parity transposition applied to (7.7) shows $F_{x(1)} + \sum_{i \in J} F_i = \sum_{i \in K \leq b-(p+2)} F_i < F_{b-(p+1)} \leq F_{x(1)}$, a contradiction. We conclude $x(1) \leq b-(p+3)$.

Since $x(i) \leq b-(p+3)$ for all i we may apply Theorem 5.6 to (7.7) with $b-(p+2)$ replacing b. We conclude that for some even $q, x(1) = (b-p-2)-q, x(3) = (b-p-2)-q+1, x(4) = (b-p-2)-q+3, \ldots, x(k-p) = (b-p-2)-1$. Define o and o' by the equations $(b-p-2)-q = b-o-4-o'$ and $b-o-4 = b-p-3$. Since b, q, and p are assumed even it follows that o and o' are odd. Combining these results with (7.4) yields (7.6).

Proof of (c): We assume to the contrary that (7.4) holds with $p \geq 1, p$ odd and derive a contradiction. For notational convenience we first assume $p \geq 3$. Then

$$0 = [F_{-x(k)} - F_{b-2} + F_{-x(k-1)} + \cdots + F_{-x(k-(p-1))}] + F_{-x(k-p)} + F_{-x(k-p-1)} + \cdots + F_{-x(3)} + F_{x(1)},$$

by (7.1),

$$\begin{aligned}
&= [F_{b-1} - F_{b-2} + (-F_{b-2} + F_{b-3}) + (-F_{b-4} + F_{b-5}) + \cdots + (-F_{b-(p-1)} + F_{b-p})] + \\
&\qquad + F_{-x(k-p)} + F_{-x(k-p-1)} + \cdots + F_{-x(3)} + F_{x(1)}, \text{ by (7.4)}, \\
&= [F_{b-3} - F_{b-4} - F_{b-6} - \cdots - F_{b-(p+1)}] + F_{-x(k-p)} + F_{-x(k-p-1)} + \cdots + F_{-x(3)} + F_{x(1)} \\
&= F_{b-(p+2)} + F_{-x(k-p)} + F_{-x(k-p-1)} + \cdots + F_{-x(3)} + F_{x(1)}, \text{ by Fibonacci Telescoping.}
\end{aligned}$$

By the assumed maximality of p in (7.4), $\{x(j) : j \leq k-p, x(j) \text{ even}\} \leq b-p-3$ for $3 \leq j \leq p$. Hence applying a parity transposition to the last equation yields $F_{b-p-2} + \sum_{i \in J} F_i = \sum_{i \in K \leq b-p-3} F_i < F_{b-p-2} (*)$. An almost identical proof applies if $p = 1$.

We may summarize the results of this section as follows:

Theorem 7.4 (The below-b-case): Suppose for some $m \geq 3$ that $b, x(1), x(3), x(4), \ldots, x(m)$ is a *prime, large, even* solution of (1.1). Suppose further that for some $i, x(i) \geq b$. Then there is a unique subscript j_0, with $x(j_0) = b$. Define $S_1 = \{x(i) : i < j_0\}$ and $k = \sup S_1$. Then either $k = 1$, and $x(k) = b-2$ or else one of (7.2), (7.5) or (7.6) must hold.

8. THE ODD-b CASE

In sections 5-7 we have completely described all *large, even, prime* solutions of (1.1). We have shown that these solutions are described by the 9 forms presented in the accident theorem. It remains to deal with the case that b is odd. We now prove a sequence of lemmas; each lemma asserts that no odd solution exists in some special case. The lemmas together exhaust all possible cases.

Lemma 8.2: There is no *large prime solution* of (1.1) with b odd, $m > 3$, and $x(i) < b$, all i.

Proof: Assume to the contrary that there is such a solution. By (1.1) $x(1) \leq b-1$ and $\{x(i) : x(i) \text{ odd}\} \leq b-2$. We now consider 3 cases.

Case $x(1) = b-1$ and $x(i) = b-2$ for some i: The sub-equation $F_b = F_{x(1)} + F_{-x(i)}$ violates primality since $m > 3$.

Case $x(1) = b-1$ and $\{x(i) : x(i)$ odd, $i \geq 3\} \leq b-4$: A parity transposition of (1.1) shows $F_b + \sum_{i \in J} F_i = F_{x(1)} + \sum_{i \in K \leq b-4} F_i < F_{b-1} + F_{b-3} (*)$.

Case $x(1) \leq b-2$ and $\{x(i) : x(i)$ odd, $i \geq 3\} \leq b-2$: The proof is almost identical to the previous case and hence is omitted.

Lemma 8.3: There is no *large prime solution* of (1.1) with b odd, $m > 3$, and $x(i) = b$, for some i.

Proof: Clear. If such a solution did exist then the sub-identity $F_b = F_{-b}$ would violate *primality.*

Lemma 8.4: There is no *large prime solution* of (1.1) with b odd, $m > 3$, and $x(m) = b + o$.

Proof: We assume to the contrary and derive a contradiction. First, $\{x(i) : x(i) \text{ odd }\} \leq b + o - 1$. Next, by Lemma 8.3 $x(i) \neq b$ for all i. Performing a parity transposition on (1.1) and applying Lemma 4.3(b) shows $F_b + F_{b+o} + \sum_{i \in J} F_i = F_{x(1)} + \sum_{i \in K \leq b+o-1} F_i < F_{b-1} + (F_{b+o} - F_b) < F_{b+o}$ a contradiction.

To deal with the final case in Lemma 8.6 we first need an additional Lemma.

Lemma 8.5: If $F_p = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-x(n)}$ with $2 < x(1) < n, 2 < x(3) < x(4) < \cdots < x(n),\ x(n)$ and p odd, $x(n) > p,\ n \geq 4$, then $x(n-1) = x(n) - 1$ and $x(n-2) < x(n) - 2$.

Proof: First we prove that $x(n-1) = x(n) - 1$. If we assume to the contrary then upon applying a parity transposition to the equation in the statement of Lemma 8.5 we conclude $F_{x(n)} < F_{x(1)} + F_{x(n)} + \sum_{i \in J} F_i < F_p + \sum_{i \in K \leq x(n)-3} F_i < F_p + F_{x(n)-2} \leq 2F_{x(n)-2} < F_{x(n)} (*)$.

Next we prove that $x(n-2) < x(n) - 2$. If we assume to the contrary then $F_{-x(n)} + F_{-x(n-1)} + F_{-x(n-2)} = 2F_{x(n)-2}$ and $\{x(i) : i \leq n-3\} \leq x(n) - 3$. Applying a parity transposition yields $2F_{x(n)-2} + F_{x(1)} + \sum_{i \in J} F_i < F_p + \sum_{i \in K \leq x(n)-3} F_i < F_p + F_{x(n)-2} \leq 2F_{x(n)-2} (*)$.

Lemma 8.6: There is no *large prime solution* of (1.1) with b odd, $m > 3$, and $x(m) = b+o+1$.

Proof: Assume $x(m) = b + o + 1$. First, applying Lemma 8.5 with $b = p,\ n = m$, yields $x(m-1) = b+o$ and $x(m-2) < b+o-1$. Substituting $F_{-x(m)} + F_{-x(m-1)} = F_{b+o+1} - F_{b+o} = F_{b+o-1}$ into (1.1) yields $F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \cdots + F_{-(b+o-1)}$.

If $(o-1) \geq 2$, we can again apply Lemma 8.5 with $b = p, n = m-1$, yielding $x(m-2) = b + o - 2$ and $x(m-3) < b + o - 3$. Consequently $F_{-(b+o-1)} + F_{-x(m-2)} = F_{-(b+o-3)}$.

This process continues inductively until we reach some r such that $x(m) = b+o+1, x(m-1) = b + o, x(m-2) = b + o - 2, \ldots, x(m-r) = b + 1$. But then by Fibonacci telescoping we have $F_{-x(m-r)} + F_{-x(m-r+1)} + \cdots + F_{-x(m-1)} + F_{-x(m)} = F_{-(b+1)} + F_{-(b+3)} + \cdots + F_{-(b+o)} + F_{-(b+o+1)} = F_b$. Since $x(1) < b$ we have not used all subscripts and therefore have violated primality. We conclude that our original assumption that $x(m) = b + o + 1$ was incorrect.

Theorem 8.7: For $m > 3$, there is no *large, odd, prime* solution of (1.1).

Proof: Clear. The hypothesii of Lemmas 8.2-8.4 and 8.6 completely exhaust all possibilities.

Remark: The proof of the accident theorem is complete.

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