

# A NEW GENERALIZATION OF THE GOLDEN RATIO

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## ABSTRACT

We propose a generalization of the golden section based on division in mean and extreme ratio. The associated integer sequences have many interesting properties.

## 1. GENERALIZED GOLDEN RATIOS

There have been many generalizations of the number known as *golden ratio* or *golden section*,  $\phi = \frac{1+\sqrt{5}}{2}$ . Examples are G.A. Moore's golden numbers [10] and S. Bradley's nearly golden sections [5] (see also [7] and [9]). A generalization that has been considered by several authors are the positive roots of  $x^{k+1} - x^k - 1 = 0$ ; see [12] and [14]. In this paper, a similar generalization is proposed. It is based on the original definition of  $\phi$ , division of a line segment in mean and extreme ratio.

Let  $G$  be a point dividing the segment  $\overline{AB}$  in parts of length  $a = |AG|$  and  $b = |GB|$ ; suppose  $a > b$ . The division is *mean and extreme* if the ratio of the larger to the smaller part equals the ratio of the whole segment to the larger part:

$$\frac{a}{b} = \frac{a+b}{a}.$$

Given a positive integer  $k$ , we consider divisions satisfying

$$\left(\frac{a}{b}\right)^k = \frac{a+b}{a}.$$

For  $k > 1$ , we have not one but two ratios:  $\varphi_k = \frac{a}{b}$  and  $\phi_k = \frac{a+b}{a} = 1 + \frac{1}{\varphi_k}$ . These numbers will be called the  $k$ -th *lower* and *upper golden ratio*, respectively. Obviously,  $(\varphi_k)^k = \phi_k$ . It is also evident that  $\varphi_k$  is a root of the polynomial  $p_k(x) = x^{k+1} - x - 1$  and  $\phi_k$  is a root of the polynomial  $P_k(x) = x(x-1)^k - 1$ .

**Proposition 1.1:** *For every positive integer  $k$ , the polynomials  $p_k(x)$  and  $P_k(x)$  have a unique positive root. If  $k$  is even this is the only real root, and if  $k$  is odd the polynomials have another negative root.*

**Proof:** The equation  $p_k(x) = 0$  can be rewritten as  $x^k - 1 = \frac{1}{x}$ . Thus, real roots correspond to intersections of the hyperbola  $y = \frac{1}{x}$  and the graph of the power function translated one unit downwards,  $y = x^k - 1$ . Similarly, real roots of  $P_k$  correspond to intersections of the hyperbola and the graph of the power function translated one unit to the right,  $y = (x-1)^k$ . The claims follow from elementary properties of the functions involved.  $\square$

Therefore,  $\varphi_k$  is the unique positive root of  $p_k$  and  $\phi_k$  is the unique positive root of  $P_k$ . The only instance when  $\varphi_k$  and  $\phi_k$  coincide is  $k = 1$ , when both are equal to the ordinary golden ratio  $\phi$ . The second lower golden ratio  $\varphi_2$  has been called *plastic number* by the Benedictine monk and architect Dom Hans van der Laan [1]. This is the smallest Pisot-Vijayaraghavan

number (see [4]). Its square,  $\phi_2$ , is also a cubic Pisot-Vijayaraghavan number. In Table 1, we list decimal approximations to the first five lower and upper golden ratios. As  $k$  grows, the lower golden ratios tend to 1 and the upper golden ratios tend to 2.

$k$	$\varphi_k$	$\phi_k$
1	1.6180339887	1.6180339887
2	1.3247179572	1.7548776662
3	1.2207440846	1.8191725134
4	1.1673039783	1.8566748839
5	1.1347241384	1.8812714616

Table 1: Lower and upper golden ratios.

**Proposition 1.2:**  $\lim_{k \rightarrow \infty} \varphi_k = 1, \lim_{k \rightarrow \infty} \phi_k = 2.$

**Proof:** By direct computation,  $p_k$  is strictly increasing on  $[1, \sqrt[k+1]{3}]$ , attains a negative value at  $x = 1$  and a positive value at  $x = \sqrt[k+1]{3}$ . Hence,  $p_k$  has a unique zero in this interval, i.e.  $\varphi_k \in (1, \sqrt[k+1]{3})$ . The proposition follows from  $\lim_{k \rightarrow \infty} \sqrt[k+1]{3} = 1$  and  $\phi_k = 1 + \frac{1}{\varphi_k}$ .  $\square$

## 2. ASSOCIATED INTEGER SEQUENCES

The connection between the golden ratio and Fibonacci numbers is well known. We can define integer sequences associated with the generalized golden ratios in a similar manner. The  $k$ -th lower Fibonacci sequence  $f_n^{(k)}$  is defined by  $f_1^{(k)} = f_2^{(k)} = \dots = f_{k+1}^{(k)} = 1$  and the linear recurrence with characteristic polynomial  $p_k$ :

$$f_n^{(k)} = f_{n-k}^{(k)} + f_{n-k-1}^{(k)}.$$

The  $k$ -th upper Fibonacci sequence  $F_n^{(k)}$  satisfies the same initial conditions and the linear recurrence with characteristic polynomial  $P_k$ . By the binomial theorem, we get

$$F_n^{(k)} = \sum_{i=1}^k \binom{k}{i} (-1)^{i+1} F_{n-i}^{(k)} + F_{n-k-1}^{(k)}.$$

Of course, both  $f_n^{(1)}$  and  $F_n^{(1)}$  are just the Fibonacci numbers. The second lower Fibonacci sequence has been called the *Padovan sequence* in [13]:

$$(f_n^{(2)}) = (1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, \dots).$$

This is sequence number A000931 in N. Sloane's *Encyclopedia of Integer Sequences* [11]. Another interesting sequence satisfying the same recurrence with different initial conditions is the

Perrin sequence (Sloane's A001608), giving a necessary condition for primality [2]. The second upper Fibonacci sequence is Sloane's A005251:

$$(F_n^{(2)}) = (1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, 616, 1081, \dots).$$

Among other combinatorial interpretations,  $F_n^{(2)}$  is the number of compositions of  $n$  without 2's [6] and the number of binary strings of length  $n - 3$  without isolated ones [3]. Notice that  $F_{n+1}^{(2)} = f_{2n-1}^{(2)}$ .

The third lower Fibonacci sequence is listed in [11] as A079398:

$$(f_n^{(3)}) = (1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 7, 8, 9, 12, 15, 17, 21, 27, 32, \dots).$$

Upper Fibonacci sequences are currently listed up to  $k = 5$ . Here are the first few values of  $F_n^{(3)}$ , Sloane's A003522:

$$(F_n^{(3)}) = (1, 1, 1, 1, 2, 5, 11, 21, 37, 64, 113, 205, 377, 693, 1266, \dots).$$

De Villiers [14] considered sequences defined by the recurrence  $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-k-1}^{(k)}$ .

When equipped with Fibonacci-like initial conditions,  $L_1^{(k)} = \dots = L_{k+1}^{(k)} = 1$ , these are the *Lamé sequences of higher order* (according to [11]). De Villiers gave a partial proof that ratios of consecutive members tend to the positive root of  $x^{k+1} - x^k - 1 = 0$ , generalizing a famous property of the Fibonacci numbers. The proof was later completed by S. Falcon [8]. Not surprisingly, ratios of consecutive members of the lower and upper Fibonacci sequences tend to the corresponding golden ratios.

**Theorem 2.1:**  $\lim_{n \rightarrow \infty} \frac{f_{n+1}^{(k)}}{f_n^{(k)}} = \varphi_k, \lim_{n \rightarrow \infty} \frac{F_{n+1}^{(k)}}{F_n^{(k)}} = \phi_k.$

**Proof:** The polynomials  $p_k, P_k$  and their derivatives are relatively prime. Therefore,  $p_k$  and  $P_k$  have  $k + 1$  distinct complex roots each and formulae for the corresponding integer sequences are of the form  $a_n = C_0 z_0^n + \dots + C_k z_k^n$ . Here,  $z_0, \dots, z_k$  are the roots of  $p_k$  or  $P_k$  and  $C_0, \dots, C_k$  are constants. The quotient of two consecutive sequence members can be expressed as

$$\frac{a_{n+1}}{a_n} = \frac{C_0 z_0^{n+1} + \dots + C_k z_k^{n+1}}{C_0 z_0^n + \dots + C_k z_k^n} = \frac{C_0 z_0 + C_1 z_1 \left(\frac{z_1}{z_0}\right)^n + \dots + C_k z_k \left(\frac{z_k}{z_0}\right)^n}{C_0 + C_1 \left(\frac{z_1}{z_0}\right)^n + \dots + C_k \left(\frac{z_k}{z_0}\right)^n}.$$

Suppose  $|z_0| > |z_i|$  for  $i = 1, \dots, k$ . Then,  $\left(\frac{z_i}{z_0}\right)^n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\frac{a_{n+1}}{a_n} \rightarrow z_0$ , provided  $C_0 \neq 0$ . Thus, it remains to be shown that the coefficients  $C_0, \dots, C_k$  are not zero and  $\varphi_k, \phi_k$  are greater than the absolute values of the remaining roots of  $p_k$  and  $P_k$ .

The coefficients  $C_0, \dots, C_k$  satisfy the system of linear equations

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_k \\ z_0^2 & z_1^2 & \cdots & z_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_0^{k+1} & z_1^{k+1} & \cdots & z_k^{k+1} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Let  $A$  be the square matrix on the left. By Cramer's rule we have

$$C_i = \frac{1}{\det A} \begin{vmatrix} z_0 & \cdots & 1 & \cdots & z_k \\ z_0^2 & \cdots & 1 & \cdots & z_k^2 \\ \vdots & & \vdots & & \vdots \\ z_0^{k+1} & \cdots & 1 & \cdots & z_k^{k+1} \end{vmatrix}.$$

The Vandermonde determinant in the numerator is not zero because the roots are all distinct and 1 is neither a root of  $p_k$  nor of  $P_k$ .

Finally, let  $z = x + iy \neq \varphi_k$  be a root of  $p_k$  and denote its absolute value by  $r = |z| = \sqrt{x^2 + y^2}$ . By Proposition 1.1,  $z$  is either the unique negative root (for odd  $k$ ) or else  $y \neq 0$ ; in both cases  $x < r$ . Taking the absolute value of  $p_k(z) = 0$  we have:

$$|z|^{k+1} = |z + 1| = \sqrt{(x + 1)^2 + y^2} < \sqrt{x^2 + y^2 + 2r + 1}.$$

Equivalently,  $r^{k+1} < \sqrt{r^2 + 2r + 1} = r + 1$ , i.e.  $p_k(r) < 0$ . The polynomial  $p_k$  is strictly increasing on  $[1, +\infty)$  and  $p_k(\varphi_k) = 0$ . Therefore,  $p_k(x) > 0$  for all  $x > \varphi_k$  and we conclude  $r < \varphi_k$ . Similarly, if  $z = x + iy \neq \phi_k$  is a root of  $P_k$ , we get

$$1 = |z| \cdot |z - 1|^k = r\sqrt{r^2 - 2x + 1}^k > r(r - 1)^k \implies P_k(r) < 0.$$

Again,  $P_k(x) > 0$  for all  $x > \phi_k$  and  $r < \phi_k$  follows. This completes the proof.  $\square$

**Corollary 2.2:**  $\lim_{n \rightarrow \infty} \frac{f_{n+k}^{(k)}}{f_n^{(k)}} = \phi_k$ .

**Proof:** By the preceding theorem, consecutive ratios of the  $k$ -th lower Fibonacci sequences tend to  $\varphi_k$  so we have

$$\frac{f_{n+k}^{(k)}}{f_n^{(k)}} = \frac{f_{n+k}^{(k)}}{f_{n+k-1}^{(k)}} \cdot \frac{f_{n+k-1}^{(k)}}{f_{n+k-2}^{(k)}} \cdots \frac{f_{n+1}^{(k)}}{f_n^{(k)}} \rightarrow \varphi_k \cdot \varphi_k \cdots \varphi_k = (\varphi_k)^k = \phi_k. \quad \square$$

Just like ordinary Fibonacci numbers, their upper “cousins” can be expressed as sums of binomial coefficients. We will need the following lemma.

**Lemma 2.3:** For any  $k \leq l \leq m$ ,  $\sum_{j=0}^k (-1)^j \binom{k}{j} \binom{m-j}{l} = \binom{m-k}{l-k}$ .

**Proof:** Let  $M$  be a set of  $m$  elements and suppose a subset of  $k$  elements is given. The right side enumerates all  $l$ -element subsets of  $M$  containing the given  $k$  elements. On the other hand,  $\binom{k}{j} \binom{m-j}{l}$  is the number of  $l$ -subsets avoiding at least  $j$  of the  $k$  given elements. The sum on the left equals the binomial coefficient on the right by inclusion-exclusion.  $\square$

**Proposition 2.4:**  $F_{n+1}^{(k)} = \sum_{i \geq 0} \binom{n-i}{k i}$

**Proof:** Obviously,  $\sum_{i \geq 0} \binom{n-i}{k i} = 1$  for all  $n \leq k$ . The recurrence for the upper Fibonacci numbers can be rewritten as

$$\sum_{j=0}^k (-1)^j \binom{k}{j} F_{n+k+1-j}^{(k)} = F_n^{(k)}.$$

By substituting appropriate sums of binomial coefficients we get

$$\sum_{j=0}^k (-1)^j \binom{k}{j} \sum_{i \geq 0} \binom{n+k-j-i}{k i} = \sum_{i \geq 0} \binom{n-1-i}{k i} = \sum_{i \geq 1} \binom{n-i}{k(i-1)}.$$

Equivalently,

$$\sum_{i \geq 1} \left[ \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{n+k-i-j}{k i} - \binom{n-i}{k(i-1)} \right] = 0.$$

The terms in the square brackets are all zero by Lemma 2.3 for  $m = n+k-i$  and  $l = k i$ . Therefore, the considered sums satisfy the the initial conditions and the recurrence for the upper Fibonacci sequence.  $\square$

Members of the Lamé sequences can also be expressed as sums of binomial coefficients [11]:

$$L_{n+1}^{(k)} = \sum_{i=0}^{\lfloor n/k \rfloor} \binom{n-ki}{i}.$$

It would be of interest to find a similar formula for the lower Fibonacci sequences and to generalize other known properties of Fibonacci numbers.

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