NONEXISTENCE OF ODD PERFECT NUMBERS OF A CERTAIN FORM

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ABSTRACT

Write $N = p^{\alpha} q_1^{2\beta_1} \cdots q_k^{2\beta_k}$, where $p, q_1, \ldots, q_k$ are distinct odd primes and $p \equiv \alpha \equiv 1 \pmod 4$. An odd perfect number, if it exists, must have this form. McDaniel proved in 1970 that N is not perfect if all β_i are congruent to $1 \pmod 3$. Hagis and McDaniel proved in 1975 that N is not perfect if all β_i are congruent to $17 \pmod{35}$. We prove that N is not perfect if all β_i are congruent to $32 \pmod{65}$. We also show that N is not perfect if all β_i are congruent to $2 \pmod 5$ and either $7|N$ or $3|N$. This is related to a result of Iannucci and Sorli, who proved in 2003 that N is not perfect if each β_i is congruent either to $2 \pmod 5$ or $1 \pmod 3$ and $3|N$.

1. INTRODUCTION

Write

$$N = p^{\alpha} q_1^{2\beta_1} \cdots q_k^{2\beta_k}, \tag{1.1}$$

where $p, q_1, \ldots, q_k$ are distinct odd primes, $\alpha, \beta_1, \ldots, \beta_k \in \mathbb{N}$, and $p \equiv \alpha \equiv 1 \pmod 4$. Euler proved that an odd perfect number, if it exists, must have the form (1.1). Let $\mathcal{O}$ denote the set of odd perfect numbers. In the case $\beta_1 = \cdots = \beta_k = \beta$, Hagis and McDaniel [3, p. 27] conjectured that $N \notin \mathcal{O}$. This conjecture was already proved for $\beta = 1$ in 1937 [7] and for $\beta = 2$ in 1941 [5]. More recently, the conjecture has been proved for some larger values of β, including $\beta = 3, 5, 6, 8, 11, 12, 14, 17, 18, 24,$ and 62 (see [1]). We now describe some infinite classes of β for which the conjecture is known to hold. Write

$$\gamma_i := 2\beta_i + 1, \quad 1 \le i \le k. \tag{1.2}$$

The assertion

$$d|\gamma_i \quad \text{for all } i \Rightarrow N \notin \mathcal{O} \tag{1.3}$$

was proved for $d = 3$ by McDaniel [6] in 1970, and for $d = 35$ by Hagis and McDaniel [3] in 1975. In particular, this proves the conjecture for the infinite classes $\beta \equiv 1 \pmod 3$ and $\beta \equiv 17 \pmod{35}$.

In Theorem 2 (see Section 3), we prove (1.3) for $d = 65$, which in particular proves the conjecture for all $\beta \equiv 32 \pmod{65}$. When d is a product of two primes > 3, the only values of d for which (1.3) is known are now $d = 35, 65$. There are no prime values $d > 3$ for which (1.3) is known.

Recently, Iannucci and Sorli [4] extended the result of McDaniel [6] by proving that

$$(3|N \text{ and } \gcd(\gamma_i, 15) > 1 \text{ for all } i) \Rightarrow N \notin \mathcal{O}. \tag{1.4}$$

(This has an important application to bounds for the number of prime factors in odd perfect numbers.) We can prove the following related results:

$$(3|N \text{ and } 7|\gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.5}$$

$$(7|N \text{ and } 5|\gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.6}$$

$$(5|N \text{ and } 77|\gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.7}$$

$$(3|N \text{ and } 143|\gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}, \tag{1.8}$$

$$(13|N \text{ and } 55|\gamma_i \text{ for all } i) \Rightarrow N \notin \mathcal{O}. \tag{1.9}$$

Of the last five assertions, we prove here only (1.6); see Theorem 1. Our proofs, like the proofs of McDaniel et. al., depend on the following result of Kanold [5]:

$$(N \in \mathcal{O} \text{ and } d|\gamma_i \text{ for all } i) \Rightarrow d^4|N. \tag{1.10}$$

2. PRELIMINARIES

Let $\sigma(n)$ denote the sum of the positive divisors of n. Assume for the purpose of contradiction that $N \in \mathcal{O}$, so that, as in [4, eq.(2)],

$$2N = \sigma(N) = \sigma(p^\alpha) \prod_{i=1}^{k} \sigma(q_i^{2\beta_i}). \tag{2.1}$$

Define, for prime q and integer $d > 1$,

$$f(q) := f_d(q) = \sigma(q^{d-1}) = (q^d - 1)/(q - 1) \tag{2.2}$$

and

$$h(q) := h_d(q) = \sigma(q^{d-1})/q^{d-1}. \tag{2.3}$$

If $d|\gamma_i$ for all i, then for all i,

$$f_d(q_i) \text{ divides } f_{\gamma_i}(q_i), \tag{2.4}$$

so $f_d(q_i)$ divides N by (2.1) - (2.2). Since α is odd,

$$(p+1)/2 \text{ divides } \sigma(p^\alpha), \tag{2.5}$$

so $(p+1)/2$ divides N by (2.1). As in [4, p. 2078], it is easily seen that for odd primes $r > q$ and integers a, b, c with $a > 1,\ c > b > 1$,

$$h_c(q) > h_b(q) > h_a(r) \geq (r+1)/r. \tag{2.6}$$

Moreover, for odd prime $u \leq p$,

$$h_a(u)(p+1)/p \geq h_a(p)(u+1)/u, \tag{2.7}$$

since $h_a(x)^{-1}(x+1)/x$ is an increasing function in x for $x > 1$.

Let S denote the set of prime divisors of N. Suppose that $d|\gamma_i$ for all i. Then by (2.1) and (2.6),

$$2 = \frac{\sigma(N)}{N} = \frac{\sigma(p^\alpha)}{p^\alpha} \prod_{i=1}^{k} h_{\gamma_i}(q_i) \geq \frac{p+1}{p} \prod_{i=1}^{k} h_d(q_i) = \frac{p+1}{p} \prod_{\substack{s\in S \\ s\neq p}} h_d(s). \tag{2.8}$$

Let T be any subset of S containing a prime u satisfying the condition that $u \leq p$ if $p \in T$. We claim that

$$\frac{p+1}{p} \prod_{\substack{s\in S \\ s\neq p}} h_d(s) \geq \frac{u+1}{u} \prod_{\substack{t\in T \\ t\neq u}} h_d(t). \tag{2.9}$$

In the case $p \notin T$, (2.9) follows because

$$\prod_{\substack{s\in S \\ s\neq p}} h_d(s) \geq \prod_{t\in T} h_d(t) \geq \frac{u+1}{u} \prod_{\substack{t\in T \\ t\neq u}} h_d(t);$$

in the case $p \in T$, (2.9) follows from (2.7).

Our objective is to find a set $T = T(d,u)$ as above such that

$$\frac{u+1}{u} \prod_{\substack{t\in T \\ t\neq u}} h_d(t) > 2. \tag{2.10}$$

In view of (2.8) - (2.9), this will provide the desired contradiction to the assumption that $N \in \mathcal{O}$.

3. THEOREMS AND PROOFS

We begin with a lemma. Recall that S is the set of prime divisors of N.

Lemma: *If $N \in \mathcal{O}$ and $13|\gamma_i$ for all i and $\gcd(p+1,21) = 1$, then $13 \in S$ and $W \subset S$, where*

$$\begin{aligned} W =&\{53, 79, 131, 157, 313, 443, 521, 547, 677, 859, 911, 937, \\ &1093, 1171, 1223, 1249, 1301, 1327, 1483, 1613, 1847\} \end{aligned}$$

is the set of primes $\equiv 1 \pmod{13}$ less than 1850.

Proof: By (1.10) with $d = 13$, we have $\mathbf{13} \in S$. (Bold font is used to keep track of primes confirmed to lie in S.)

A list of primes

$$r_1, r_2, \ldots, r_n \tag{3.1}$$

is called a *d-chain* (or simply a *chain*) if $r_1 \in S$ and $r_{i+1} | f_d(r_i)$ for each $i < n$, where f_d is defined in (2.2). In this proof, we take $f = f_d$ with $d = 13$. If $r_i \neq p$ for each $i < n$, then every prime in the chain (3.1) lies in S, by (2.4). An example of a chain is

$$13, 264031, (882..981), \mathbf{79}. \tag{3.2}$$

Here (882..981) is a 64-digit prime whose center digits can be easily retrieved by factoring $f(264031)$. By hypothesis, the first and third primes in (3.2) cannot be p, because they are $\equiv 6 \pmod 7$. The second and fourth primes cannot be p since they are $\equiv 3 \pmod 4$. We know $13 \in S$, so $264031 \in S$ because $264031 | f(13)$. Similarly, $(882..981) \in S$ since $(882..981) | f(264031)$. Finally, $79 | f((882..981))$, so the chain (3.2) confirms that $79 \in S$.

None of the following chains can have p preceding its terminal prime r_n, and so each chain confirms that r_n (in bold) lies in S:

$$\begin{aligned}
&13, \mathbf{53};\\
&13, 264031, (882..981), \mathbf{157};\\
&79, (551..681), \mathbf{1249};\\
&79, (551..681), 50909, \mathbf{499903};\\
&499903, \mathbf{1483};\\
&499903, 32579, (\mathbf{313} \text{ and } \mathbf{937});\\
&937, \mathbf{599};\\
&599, 847683(\mathbf{443} \text{ and } \mathbf{1613});\\
&599, 45137, 6397, (\mathbf{677} \text{ and } \mathbf{911});\\
&937, (111..851), 14561, \mathbf{42304159};\\
&42304159, \mathbf{3251};\\
&42304159, (766..419), (46073), (976..861), \mathbf{859};\\
&3251, \mathbf{131};\\
&1483, (301..587), \mathbf{1223};\\
&1223, 920011, \mathbf{2081};\\
&2081, (\mathbf{547} \text{ and } \mathbf{1171});\\
&157, (281..937), 5669, 168247, (395..237), \mathbf{1327};\\
&859, (183..471), \mathbf{2029};\\
&499903, 32579, (\mathbf{468}..\mathbf{021}).
\end{aligned}$$

Next consider the pair of chains

$$\begin{cases} 313, (240..891), 9907, 1847;\\ 1249, (555..427), \mathbf{1847}. \end{cases}$$

The two chains in the pair have no common primes except the terminal prime 1847. Thus, while p might precede 1847 somewhere in one chain or the other, p cannot precede 1847 in both chains. Hence (at least) one chain in the pair does not have an occurrence of p preceding 1847, and that chain confirms that $1847 \in S$. We now can form the single chains

$$\begin{aligned}&1847, \mathbf{521};\\&521, (317..359), \mathbf{1951};\\&1951, (193..027), 4759, \mathbf{1301}.\end{aligned}$$

It remains to show that $1093 \in S$. This is accomplished with the following pair of chains:

$$\left\{\begin{array}{l}2029, 65677, 18038593, 1093;\\(468..021), 138581, (648..279), (112..139), 1873, (110..713), (582..641),\\\quad(578..461), \mathbf{1093}.\end{array}\right.$$

$\square$

Theorem 1: *Suppose that* $5|\gamma_i$ *for all* i*, and* $N \in \mathcal{O}$*. Then* $\gcd(N, 21) = 1$ *and* $p \equiv 1 \pmod{12}$*.*

Proof: By (1.10) with $d = 5$, we have $5 \in S$.

Suppose for the purpose of contradiction that $p \equiv 2 \pmod 3$. Then by (2.5), $3 \in S$. As in (2.2), write $f = f_d$ with $d = 5$. Since $f(3) = 11^2$, (2.4) implies that $11 \in S$. Since $5|\gamma_i$ for all i and $5^4|N$ by (1.10), then, in the notation of (2.3) with $d = 5$, we obtain the contradiction

$$2 = \sigma(N)/N > h(3)h(5)h(11) > 2.05\ . \tag{3.3}$$

This proves that $p \equiv 1 \pmod{12}$.

We have seen that $5 \in S$. We now confirm additional primes in S by using d-chains as in the Lemma, but with $d = 5$ instead of $d = 13$. The chains

$$\begin{aligned}&5, (\mathbf{11} \text{ and } \mathbf{71});\\&11, 3221, (195..931), \mathbf{41};\end{aligned}$$

confirm that $11, 71$, and 41 lie in S, since neither 5 nor 3221 can equal p (as $p \equiv 1 \pmod{12}$). Employing many such chains, we can construct a large set Y of primes in S consisting of 5 together with most of the primes $\equiv 1 \pmod 5$ which are $< 10^4$. The set Y and the long list of chains used to construct Y may be found at [2].

Suppose that $7|N$. With $T = Y \cup \{7\}$, we arrive at the contradiction (2.10) with $u = 61, d = 5$. Thus $7 \nmid N$. The same argument shows that $3 \nmid N$ (alternatively, $3 \nmid N$ follows from (1.4)). This completes the proof of Theorem 1. $\square$

Theorem 2: *If* $65|\gamma_i$ *for all* i*, then* $N \notin \mathcal{O}$*.*

Proof: Assume for the purpose of contradiction that $65|\gamma_i$ for all i and $N \in \mathcal{O}$. From (1.10), we know that $13 \in S$. Let Y be as in the proof of Theorem 1, and let W be as defined in the Lemma. In view of Theorem 1, the hypotheses of the Lemma are satisfied, and so $Y \cup W \subset S$. With

$$T = Y \cup W \cup \{13\},$$

we obtain the desired contradiction (2.10) with $u = 61, d = 65$. This completes the proof of Theorem 2. □

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