SOME COMPUTATIONAL FORMULAS FOR NÖRLUND NUMBERS

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ABSTRACT

In this paper we investigate the relation for the Bernoulli numbers of higher order, the Stirling numbers and associated Stirling numbers, and establish some computational formulas for the Nörlund numbers.

1. INTRODUCTION AND RESULTS

The Nörlund numbers N_n and the Bernoulli polynomials $B_n^{(k)}(x)$ of order k are defined, respectively, by (see [1], [4], [6])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} N_n \frac{t^n}{n!}, \tag{1.1}$$

and

$$(\frac{t}{e^t-1})^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.2}$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers. By (1.1) and (1.2), we can get (see [4], [6])

$$N_n = B_n^{(n)}. \tag{1.3}$$

Stirling numbers of the first kind $s(n,k)$ can be defined by means of (see [1],[3],[5])

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k, \tag{1.4}$$

or by the generating function

$$(\log(1+x))^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}. \tag{1.5}$$

It follows from (1.4) or (1.5) that

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{1.6}$$

with $s(n,0) = 0 (n>0), s(n,n) = 1, s(n,1) = (-1)^{n-1}(n-1)!(n>0), s(n,k) = 0 (k>n$ or $k<0)$.

Stirling numbers of the second kind $S(n,k)$ can be defined by means of (see [1],[3],[5])

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)(x-2)\cdots(x-k+1), \tag{1.7}$$

or by the generating function

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}. \tag{1.8}$$

It follows from (1.7) or (1.8) that

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{1.9}$$

with $S(n,0) = 0 (n > 0), S(n,n) = 1, S(n,1) = 1 (n > 0), S(n,k) = 0 (k > n \text{ or } k < 0)$.

Associated Stirling numbers of the first kind $d(n,k)$ and associated Stirling numbers of the second kind $b(n,k)$ are defined, respectively, by (see [1],[3])

$$(\log(1+x) - x)^k = k! \sum_{n=2k}^{\infty} (-1)^{n-k} d(n,k) \frac{x^n}{n!}, \tag{1.10}$$

and

$$(e^x - 1 - x)^k = k! \sum_{n=2k}^{\infty} b(n,k) \frac{x^n}{n!}. \tag{1.11}$$

It follows from (1.10) and (1.11) that

$$d(n,k) = (n-1)d(n-2,k-1) + (n-1)d(n-1,k), \tag{1.12}$$

with $d(n,0) = 0 (n > 0), d(0,0) = 1, d(n,1) = (n-1)! (n > 1), d(n,k) = 0 (2k > n \text{ or } k < 0)$ and

$$b(n,k) = (n-1)b(n-2,k-1) + kb(n-1,k), \tag{1.13}$$

with $b(n,0) = 0 (n > 0), b(0,0) = 1, b(n,1) = 1 (n > 1), b(n,k) = 0 (2k > n \text{ or } k < 0)$.

In [2], F. T. Howard obtained the relationships between Nörlund Numbers N_n and Stirling numbers of the first kind $s(n,k)$,

$$N_n = \sum_{k=0}^{n} \frac{(-1)^k s(n,k)}{k+1}. \tag{1.14}$$

The main purpose of this paper is that to prove some computational formulas for Nörlund numbers. That is, we shall prove the following main conclusion.

Theorem 1: Let $n \geq 1$ be integers, then

$$N_n = n \cdot n! \sum_{k=0}^{n-1} (-1)^{n-1-k} \frac{(k-1)! d(n+k,k)}{(n+k)!}. \tag{1.15}$$

Theorem 2: Let $n \geq 1$ be integers, then

$$N_n = \sum_{k=0}^{n} \frac{n! k!}{(n+k)!} s(n+k,n) S(n,k). \tag{1.16}$$

Theorem 3: Let $n \geq 1$ be integers, then

$$N_n = \sum_{k=0}^{n} (-1)^k \frac{n}{n+k} \binom{2n}{n+k} S(n+k,k). \tag{1.17}$$

Theorem 4: Let $n \geq 1$ be integers, then

$$N_n = \sum_{k=0}^{n} (-1)^k \frac{n}{n+k} b(n+k,k). \tag{1.18}$$

2. PROOF OF THE THEOREMS

Proof of Theorem 1: By (1.1), we have

$$\sum_{n=0}^{\infty}\frac{N_n}{n}\frac{t^n}{n!}=\log\frac{\log(1+t)}{t}. \tag{2.1}$$

Therefore, By (2.1) and (1.10), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{N_n}{n}\frac{t^n}{n!}&=\log\frac{\log(1+t)}{t}=\log(1+\frac{\log(1+t)-t}{t})=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}(\frac{\log(1+t)-t}{t})^k\\&=\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!\sum_{n=2k}^{\infty}(-1)^{n-k}d(n,k)\frac{t^{n-k}}{n!}\\&=\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!\sum_{n=k}^{\infty}(-1)^{n}d(n+k,k)\frac{t^{n}}{(n+k)!}\\&=\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n-k-1}(k-1)!d(n+k,k)\frac{t^{n}}{(n+k)!}.\end{aligned} \tag{2.2}$$

Comparing the coefficient of t^n on both sides of (2.2), we get

$$N_n=n\cdot n!\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(k-1)!d(n+k,k)}{(n+k)!}. \tag{2.3}$$

This completes the proof of Theorem 1.

Proof of Theorem 2: By (1.2), (1.5) and (1.8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}&=(\frac{t}{e^t-1})^k=(\frac{\log(1+(e^t-1))}{e^t-1})^k=k!\sum_{j=k}^{\infty}s(j,k)\frac{(e^t-1)^{j-k}}{j!}\\&=k!\sum_{j=k}^{\infty}\frac{s(j,k)}{j!}(j-k)!\sum_{n=j-k}^{\infty}S(n,j-k)\frac{t^n}{n!}=k!\sum_{j=0}^{\infty}\frac{s(j+k,k)}{(j+k)!}j!\sum_{n=j}^{\infty}S(n,j)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{k!j!}{(k+j)!}s(k+j,k)S(n,j)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Comparing the coefficient of t^n on both sides of (2.4), we get

$$B_n^{(k)}=\sum_{j=0}^{n}\frac{k!j!}{(k+j)!}s(k+j,k)S(n,j). \tag{2.5}$$

Therefore,

$$N_n=B_n^{(n)}=\sum_{j=0}^{n}\frac{n!j!}{(n+j)!}s(n+j,n)S(n,j). \tag{2.6}$$

This completes the proof of Theorem 2.

Proof of Theorem 3: By (1.2) and (1.11), we have

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} = (\frac{t}{e^t-1})^k = (\frac{1}{1+(e^t-1-t)t^{-1}})^k = \sum_{i=0}^{\infty}(-1)^i \binom{k+i-1}{i} (\frac{e^t-1}{t}-1)^i$$

$$= \sum_{i=0}^{\infty}(-1)^i \binom{k+i-1}{i} \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}(\frac{e^t-1}{t})^j$$

$$= \sum_{i=0}^{\infty}(-1)^i \binom{k+i-1}{i} \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j} j! \sum_{n=j}^{\infty} S(n,j)\frac{t^{n-j}}{n!}$$

$$= \sum_{i=0}^{\infty} \binom{k+i-1}{i} \sum_{j=0}^{i}(-1)^{j}\binom{i}{j} j! \sum_{n=0}^{\infty} S(n+j,j)\frac{t^{n}}{(n+j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \binom{k+i-1}{i} \sum_{j=0}^{i}(-1)^{j}\binom{i}{j} j! S(n+j,j)\frac{t^{n}}{(n+j)!}. \tag{2.7}$$

Comparing the coefficient of t^n on both sides of (2.7), we get

$$B_n^{(k)} = \sum_{i=0}^{n} \binom{k+i-1}{i} \sum_{j=0}^{i}(-1)^{j}\binom{i}{j} \frac{n!j!}{(n+j)!} S(n+j,j)$$

$$= \sum_{j=0}^{n}(-1)^{j} \frac{n!j!}{(n+j)!} S(n+j,j) \sum_{i=j}^{n} \binom{k+i-1}{i}\binom{i}{j}$$

$$= \sum_{j=0}^{n}(-1)^{j} \frac{n!j!}{(n+j)!} S(n+j,j) \binom{k+j-1}{j}\binom{n+k}{k+j}. \tag{2.8}$$

Therefore,

$$N_n = B_n^{(n)} = \sum_{j=0}^{n}(-1)^{j} \frac{n!j!}{(n+j)!} S(n+j,j) \binom{n+j-1}{j}\binom{2n}{n+j}$$

$$= \sum_{j=0}^{n}(-1)^{j} \frac{n}{(n+j)} \binom{2n}{n+j} S(n+j,j). \tag{2.9}$$

This completes the proof of Theorem 3.

Proof of Theorem 4: By (1.2) and (1.11), we have

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} = (\frac{t}{e^t-1})^k = (\frac{1}{1+(e^t-1-t)t^{-1}})^k = \sum_{j=0}^{\infty}(-1)^j \binom{k+j-1}{j} (e^t-1-t)^j t^{-j}$$

$$= \sum_{j=0}^{\infty}(-1)^j \binom{k+j-1}{j} j! \sum_{n=2j}^{\infty} b(n,j)\frac{t^{n-j}}{n!} = \sum_{j=0}^{\infty}(-1)^j \binom{k+j-1}{j} j! \sum_{n=j}^{\infty} b(n+j,j)\frac{t^{n}}{(n+j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j j! \binom{k+j-1}{j} b(n+j,j)\frac{t^{n}}{(n+j)!}. \tag{2.10}$$

Comparing the coefficient of t^n on both sides of (2.10), we get

$$B_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{n!j!}{(n+j)!} \binom{k+j-1}{j} b(n+j,j). \tag{2.11}$$

Therefore,

$$N_n = B_n^{(n)} = \sum_{j=0}^{n} (-1)^j \frac{n!j!}{(n+j)!} \binom{n+j-1}{j} b(n+j,j) = \sum_{j=0}^{n} (-1)^j \frac{n}{n+j} b(n+j,j). \tag{2.12}$$

This completes the proof of Theorem 4.

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