SEVERAL IDENTITIES INVOLVING THE FIBONACCI NUMBERS AND LUCAS NUMBERS

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(Submitted September 2006-Final Revision May 2007)

ABSTRACT

In this paper, we define the Chebyshev polynomials representation of x^n, then use this expression to study the Fibonacci numbers and Lucas numbers by the elementary method, and get some beautiful identities about Fibonacci numbers and Lucas numbers.

1. INTRODUCTION

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}\{n = 0, 1, 2, ...\}$ are defined by the second-order linear recurrence sequence

$$F_{n+2} = F_{n+1} + F_n$$

$$L_{n+2} = L_{n+1} + L_n$$

where $n \geq 0,\ F_0 = 0,\ F_1 = 1,\ L_0 = 2$ and $L_1 = 1$. These sequences play a very important role in the study of the theory and application of mathematics. Therefore, the various properties of F_n and L_n have been investigated by many authors. For example, R.L. Duncan[2] and L.Kuipers[3] have proved that $(\log F_n)$ is uniformly distributed mod 1. Neville Robbins[4] has studied the Fibonacci numbers of the forms $px^2 \pm 1,\ px^3 \pm 1$, where p is a prime. The second author Zhang Wenpeng[5] has obtained the general formulas involving F_n and L_n

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{m(a_1+1)}F_{m(a_2+1)}\dots F_{m(a_{k+1}+1)} = (-1)^{mn}\frac{F_m^{k+1}}{2^k k!}U_{n+k}^{(k)}\left(\frac{i^m}{2}L_m\right),$$

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n+k+1} L_{ma_1}L_{ma_2}\dots L_{ma_{k+1}}$$

$$= (-1)^{m(n+k+1)}\frac{2}{k!}\sum_{h=0}^{k+1}\left(\frac{i^{m+2}}{2}L_m\right)^h \frac{(k+1)!}{h!(k+1-h)!}U_{n+2k+1-h}^{(k)}\left(\frac{i^m}{2}L_m\right),$$

where k, m are any positive integers, $n, a_1, a_2, \dots, a_{k+1}$ are nonnegative integers and i is the square root of -1.

In this paper, we study the common formulas for x^n being represented by Chebyshev polynomials, then get two identities about the Fibonacci numbers and Lucas numbers. That is, we shall prove the following.

Theorem 1: For any nonnegative integer n and positive integer m, we have the identities

$$L_m^{2n} = (-1)^{mn}\frac{(2n)!}{(n!)^2} + (2n)!\sum_{k=1}^{n}\frac{(-1)^{m(n-k)}}{(n-k)!(n+k)!}L_{2km};$$

$$L_m^{2n+1} = (2n+1)!\sum_{k=0}^{n}\frac{(-1)^{m(n-k)}}{(n-k)!(n+k+1)!}L_{(2k+1)m}.$$

Theorem 2: For any nonnegative integer n and positive integer m, we have the identities

$$L_m^{2n} = \frac{(2n)!}{F_m}\sum_{k=0}^{n}\frac{(-1)^{m(n-k)}(2k+1)}{(n-k)!(n+k+1)!}F_{(2k+1)m};$$

$$L_m^{2n+1} = \frac{2(2n+1)!}{F_m}\sum_{k=0}^{n}\frac{(-1)^{m(n-k)}(k+1)}{(n-k)!(n+k+2)!}F_{2(k+1)m}.$$

2. SOME LEMMAS

In this section, we shall give several lemmas which are necessary in the proofs of the theorems. First we need two exact expressions on Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)\,(n = 0, 1, \cdots)$

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right],$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right].$$

There is another way of defining Chebyshev polynomials as follows

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x),$$

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x),$$

for $n \geq 0,\ T_0(x) = x,\ T_1(x) = x,\ U_0(x) = 1$ and $U_1(x) = 2x$.

Furthermore, we shall have the following lemmas.

Lemma 1: For any positive integers m and n, we have the identities

$$T_n(T_m(x)) = T_{mn}(x),$$

$$U_n(T_m(x)) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)}.$$

Proof: See Ref. [5].

Lemma 2: Let i be the square root of -1, m and n any positive integers, then we have the identities

$$U_n(\frac{i}{2}) = i^n F_{n+1},$$

$$T_n(\frac{i}{2}) = \frac{i^n}{2} L_n,$$

$$T_n(T_m(\frac{i}{2})) = \frac{i^{mn}}{2} L_{mn},$$

$$U_n(T_m(\frac{i}{2})) = i^{mn} \frac{F_{m(n+1)}}{F_m}.$$

Proof: It is easy to see that $U_n(\frac{i}{2}) = i^n F_{n+1}$, $T_n(\frac{i}{2}) = \frac{i^n}{2} L_n$, then according to Lemma 1, we can see the last two formulas clearly. This proves Lemma 2.

Lemma 3: For any nonnegative integer n, let

$$x^n \equiv \frac{1}{2} a_{n0} T_0(x) + \sum_{k=1}^{\infty} a_{nk} T_k(x) \tag{1}$$

and

$$x^n \equiv \sum_{k=0}^{\infty} b_{nk} U_k(x), \tag{2}$$

then we have

$$a_{nk} = \begin{cases} \frac{2n!}{(n-k)!!(n+k)!!}, & n \geq k,\ n+k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

$$b_{nk} = \begin{cases} \frac{2(k+1)n!}{(n-k)!!(n+k+2)!!}, & n \geq k,\ n+k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Proof: As we know, Chebyshev polynomials have a lot of properties (See Ref. [1]) such as

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n > 0, \\ \pi, & m = n = 0; \end{cases} \tag{5}$$

$$T_n(\cos\theta) = \cos n\theta; \tag{6}$$

$$\int_{-1}^{1} \sqrt{1-x^2}U_m(x)U_n(x)dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n > 0, \\ \pi, & m = n = 0; \end{cases} \tag{7}$$

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{8}$$

For any nonnegative integer m, first multiply $\frac{T_m(x)}{\sqrt{1-x^2}}$ to the two sides of (1), then integrate it from -1 to 1, finally, applying the property (5), we can get

$$\begin{aligned} \int_{-1}^{1} \frac{x^n T_m(x)}{\sqrt{1-x^2}}dx &= \frac{1}{2}a_{n0}\int_{-1}^{1} \frac{T_m(x)T_0(x)}{\sqrt{1-x^2}}dx + \sum_{k=1}^{\infty} a_{nk}\int_{-1}^{1} \frac{T_m(x)T_k(x)}{\sqrt{1-x^2}}dx \\ &= \frac{\pi}{2}a_{nm}, \qquad (m = 0, 1, 2, \cdots) \end{aligned}$$

so,

$$a_{nm} = \frac{2}{\pi}\int_{-1}^{1} \frac{x^n T_m(x)}{\sqrt{1-x^2}}dx.$$

Let $x = \cos t$ and according to the property (6), we have

$$\begin{aligned} a_{nm} &= \frac{2}{\pi}\int_0^{\pi} \cos^n t \cos mt \ dt \\ &= \frac{2}{\pi}\int_0^{\pi} \cos^n t \left(\cos(m-1)t\cos t - \sin(m-1)t\sin t\right) \ dt \\ &= \frac{2}{\pi}\int_0^{\pi} \cos^{n+1} t \ \cos(m-1)t \ dt - \frac{2}{\pi}\int_0^{\pi} \cos^n t \ \sin(m-1)t \ \sin t \ dt \\ &= a_{n+1m-1} + \frac{2}{\pi}\frac{1}{n+1}\int_0^{\pi} \sin(m-1)t \ d(\cos^{n+1} t) \\ &= a_{n+1m-1} + \frac{2}{\pi}\frac{1}{n+1}\cos^{n+1} t \ \sin(m-1)t\Big|_0^{\pi} - \frac{2}{\pi}\frac{m-1}{n+1}\int_0^{\pi} \cos^{n+1} t \ \cos(m-1)t \ dt \\ &= a_{n+1m-1} - \frac{m-1}{n+1}a_{n+1m-1} \\ &= \frac{n-m+2}{n+1}a_{n+1m-1} \\ &= \frac{n-m+2}{n+1}\frac{n-m+4}{n+2}a_{n+2m-2} \\ &= \cdots \\ &= \frac{n-m+2}{n+1}\frac{n-m+4}{n+2}\cdots\frac{n-m-2m}{n+m}a_{n+m0}. \end{aligned}$$

But

$$a_{n+m0} = \frac{2}{\pi}\int_0^{\pi} \cos^{n+m} t \ dt$$

$$= \begin{cases} \frac{2(n+m-1)!!}{(n+m)!!}, & n+m \text{ is even,} \\ 0, & n+m \text{ is odd.} \end{cases}$$

So

$$a_{nm} = \begin{cases} \frac{\frac{(n+m)!!}{(n-m)!!}}{\frac{(n+m)!}{n!}} \frac{2(n+m-1)!!}{(n+m)!!} = \frac{2n!}{(n-m)!!(n+m)!!}, & n \geq m,\ n+m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $m = k$, we immediately get formula (3).

In the similar way, for any nonnegative integer m, first multiply $\sqrt{1-x^2}U_m(x)$ to the two sides of (2), then integrate from -1 to 1, finally, applying the property (7), we can get

$$\int_{-1}^{1} \sqrt{1-x^2}x^n U_m(x)\ dx = \sum_{k=0}^{\infty} b_{nk} \int_{-1}^{1} \sqrt{1-x^2}U_m(x)U_k(x)\ dx$$

$$= \frac{\pi}{2} b_{nm}, \qquad (m = 0, 1, 2, \cdots)$$

let $m = k$, then we have

$$b_{nk} = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x^2}x^n U_k(x)\ dx.$$

Let $x = \cos t$ and according to the property (8), we have

$$\begin{aligned} b_{nk} &= \frac{2}{\pi}\int_0^{\pi} \cos^n t \sin(k+1)t \sin t\ dt \\ &= \frac{2}{\pi}\frac{-1}{n+1}\int_0^{\pi} \sin(k+1)t\ d(\cos^{n+1} t) \\ &= \frac{2}{\pi}\frac{-1}{n+1}\left(\cos^{n+1} t\ \sin(k+1)t\Big|_0^{\pi} - (k+1)\int_0^{\pi} \cos^{n+1} t\ \cos(k+1)t\ dt\right) \\ &= \frac{2}{\pi}\frac{k+1}{n+1}\int_0^{\pi} \cos^{n+1} t\ \cos(k+1)t\ dt \\ &= \frac{k+1}{n+1} a_{n+1k+1}. \end{aligned}$$

According to the formula (3), we can get

$$b_{nk} = \begin{cases} \frac{2(k+1)n!}{(n-k)!!(n+k)!!}, & n \geq k,\ n+k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we get formula (4). This proves Lemma 3.

Lemma 4: For any nonnegative integer n, we also have the expressions of x^n in the following forms

$$x^{2n} = \frac{(2n)!}{4^n(n!)^2}T_0(x) + \frac{2(2n)!}{4^n}\sum_{k=1}^{n}\frac{1}{(n-k)!(n+k)!}T_{2k}(x)$$

$$= \frac{(2n)!}{4^n}\sum_{k=0}^{n}\frac{2k+1}{(n-k)!(n+k+1)!}U_{2k}(x),$$

$$x^{2n+1} = \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}T_{2k+1}(x)$$

$$= \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{k+1}{(n-k)!(n+k+2)!}U_{2k+1}(x).$$

Proof: This follows directly from Lemma 3.

3. PROOF OF THEOREMS

In this section, we shall complete the proofs of theorems. Firstly we prove Theorem 1. According to Lemma 4, let $x = T_m(x)$, then we have

$$T_m^{2n}(x) = \frac{(2n)!}{4^n(n!)^2}T_0(T_m(x)) + \frac{2(2n)!}{4^n}\sum_{k=1}^{n}\frac{1}{(n-k)!(n+k)!}T_{2k}(T_m(x)),$$

$$T_m^{2n+1}(x) = \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}T_{2k+1}(T_m(x)),$$

and let $x = \frac{i}{2}$, according to Lemma 2, we get

$$\frac{i^{2mn}}{2^{2n}}L_m^{2n} = \frac{(2n)!}{4^n(n!)^2} + \frac{2(2n)!}{4^n}\sum_{k=1}^{n}\frac{1}{(n-k)!(n+k)!}\frac{i^{2mk}}{2}L_{2mk},$$

$$\frac{i^{(2n+1)m}}{2^{2n+1}}L_m^{2n+1} = \frac{(2n+1)!}{2^{2n}}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}\frac{i^{(2k+1)m}}{2}L_{m(2k+1)}.$$

That is,

$$L_m^{2n} = (-1)^{mn}\frac{(2n)!}{(n!)^2} + (2n)!\sum_{k=1}^{n}\frac{(-1)^{m(k-n)}}{(n-k)!(n+k)!}L_{2km},$$

$$L_m^{2n+1} = (2n+1)!\sum_{k=0}^{n}\frac{(-1)^{m(k-n)}}{(n-k)!(n+k+1)!}L_{m(2k+1)}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. According to Lemma 4, we have

$$T_m^{2n}(x) = \frac{(2n)!}{4^n}\sum_{k=0}^{n}\frac{2k+1}{(n-k)!(n+k+1)!}U_{2k}(T_m(x)),$$

$$T_m^{2n+1}(x) = \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{k+1}{(n-k)!(n+k+2)!}U_{2k+1}(T_m(x)),$$

and let $x = \frac{i}{2}$, according to Lemma 2, we have

$$\frac{i^{2mn}}{2^{2n}}L_m^{2n} = \frac{(2n)!}{4^n}\sum_{k=0}^{n}\frac{2k+1}{(n-k)!(n+k+1)!}i^{2mk}\frac{F_{m(2k+1)}}{F_m},$$

$$\frac{i^{m(2n+1)}}{2^{2n+1}}L_m^{2n+1} = \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{k+1}{(n-k)!(n+k+2)!}i^{(2k+1)m}\frac{F_{m(2k+2)}}{F_m}.$$

That is

$$L_m^{2n} = (2n)!\sum_{k=0}^{n}\frac{(-1)^{m(k-n)}(2k+1)}{(n-k)!(n+k+1)!}\frac{F_{m(2k+1)}}{F_m},$$

$$L_m^{2n+1} = 2(2n+1)!\sum_{k=0}^{n}\frac{(-1)^{m(k-n)}(k+1)}{(n-k)!(n+k+2)!}\frac{F_{2m(k+1)}}{F_m}.$$

This proves Theorem 2.

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AMS Classification Numbers: 11B39, 11B83

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