SOME FORMULAE FOR THE FIBONACCI NUMBERS

Brian Curtin
Department of Mathematics, University of South Florida, 4202 E. Fowler Ave. PHY114, Tampa, FL 33620
e-mail: bcurtin@math.usf.edu

Ena Salter
Department of Mathematics, Manatee Community College,
8000 South Tamiami Tr., Office #617, Venice, FL 34293
e-mail: saltere@mccfl.edu

David Stone
Mathematical Sciences, Georgia Southern University, PO Box 8093, Statesboro, GA 30460-8093
e-mail: dstone@GeorgiaSouthern.edu

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ABSTRACT

Using elementary methods, we derive formulae for the shifted summations $\sum_{j=0}^{d-1} F_{n+j}F_{m+j}$, $\sum_{j=0}^{d-1} L_{n+j}L_{m+j}$, and $\sum_{j=0}^{d-1} F_{n+j}L_{m+j}$ and the shifted convolutions $\sum_{j=0}^{d-1} F_{n+j}F_{d-m-j}$, $\sum_{j=0}^{d-1} L_{n+j}L_{d-m-j}$, and $\sum_{j=0}^{d-1} F_{n+j}L_{d-m-j}$ for positive integers d and arbitrary integers n and m.

1. INTRODUCTION

We derive the following identities involving the Fibonacci numbers F_n and the Lucas numbers L_n. For all positive integers d and for all integers n and m,

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j} = \begin{cases} F_dF_{n+m+d-1} & \text{if } d \text{ is even,} \\ \frac{1}{5}\left(L_dL_{n+m+d-1} - (-1)^nL_{m-n}\right) & \text{if } d \text{ is odd,} \end{cases} \tag{1}$$

$$\sum_{j=0}^{d-1} L_{n+j}L_{m+j} = \begin{cases} 5F_dF_{n+m+d-1} & \text{if } d \text{ is even,} \\ L_dL_{n+m+d-1} + (-1)^nL_{m-n} & \text{if } d \text{ is odd,} \end{cases} \tag{2}$$

$$\sum_{j=0}^{d-1} F_{n+j}L_{m+j} = \begin{cases} F_dL_{n+m+d-1} & \text{if } d \text{ is even,} \\ L_dF_{n+m+d-1} + (-1)^{n+1}F_{m-n} & \text{if } d \text{ is odd,} \end{cases} \tag{3}$$

$$\sum_{j=0}^{d-1} F_{n+j}F_{d+m-j-1} = \frac{1}{5}\left(dL_{n+m+d-1} - (-1)^nL_{m-n}F_d\right), \tag{4}$$

$$\sum_{j=0}^{d-1} L_{n+j}L_{d+m-j-1} = dL_{n+m+d-1} + (-1)^nL_{m-n}F_d, \tag{5}$$

$$\sum_{j=0}^{d-1} F_{n+j}L_{d+m-j-1} = dF_{n+m+d-1} + (-1)^{n+1}F_{m-n}F_d. \tag{6}$$

We also present a number of interesting consequences of these formulae. We wish to emphasize three aspects of this paper which we find particularly satisfying. First, our methods are elementary–we believe that this paper is accessible to anyone who has completed an elementary linear algebra course. Indeed, we use little more than the Binet formulae for the Fibonacci and Lucas numbers, the sum of a finite geometric series, and linearity of the dot product. In fact, the standard linear algebraic derivation [2] of the Binet formulas fits well into the framework of Fibonacci vectors. Second, our methods are productive. That is to say, carrying out the proofs produces the formulae, in contrast to many inductions which require prior knowledge of the formula to be proved. Finally, our results concerning Fibonacci and Lucas numbers closely parallel one another. This further supports our faith in the orderliness of the subject.

We note that special cases of (1)–(3) appear in [3] (modulo a few identities) as Equations (4)–(7). For example, Equation (4) in [3] is

$$\sum_{j=0}^{d-1} F_{1+j}F_{m+1+j} = \begin{cases} F_d F_{m+d} & \text{if } d \text{ is even,} \\ F_d F_{m+d} - F_{m-1} & \text{if } d \text{ is odd} \end{cases} \tag{7}$$

The left side of (7) is the same as that of (1) with $n = 1$. When d is even, the corresponding right sides are exactly the same, and when d is odd, a few elementary identities, such as $L_x L_y - 5F_x F_y = 2(-1)^y L_{x-y}$, transform one right side into the other. Conversely, subtracting from (7) the same equation with $d = n$ leaves a sum like that on the left side of (1). Again various identities show that the corresponding right sides are equal. We note that our methods are quite different from those of [3] and have also led us to (4)–(6). We also believe that our methods both open up interesting avenues of investigation and can be pushed further than they have been here.

2. SOME VECTORS

In this section, we introduce some vectors. For the rest of this paper, we fix the length of all vectors to be some positive integer d. We suppress the dependence of the vectors on d in our notation.

Definition 2.1: For all integers n, define

$$\overrightarrow{f_n} = \begin{bmatrix} F_n \\ F_{n+1} \\ \vdots \\ F_{n+d-1} \end{bmatrix} \quad \text{and} \quad \overrightarrow{l_n} = \begin{bmatrix} L_n \\ L_{n+1} \\ \vdots \\ L_{n+d-1} \end{bmatrix}.$$

We refer to $\overrightarrow{f_n}$ and $\overrightarrow{l_n}$ as the *n-th Fibonacci and Lucas vectors (of length d)*, respectively.

The following two vectors play a fundamental role in the study of Fibonacci and Lucas vectors.

Definition 2.2: Define

$$\overrightarrow{a} = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{d-1} \end{bmatrix} \text{ and } \overrightarrow{b} = \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \\ \beta^{d-1} \end{bmatrix}.$$

The key facts relating $\overrightarrow{a}$ and $\overrightarrow{b}$ to the Fibonacci and Lucas vectors are the following generalizations of the Binet formulae.

Theorem 2.3: *For all integers n,*

$$\overrightarrow{f_n} = \frac{1}{\alpha - \beta}\left(\alpha^n \overrightarrow{a} - \beta^n \overrightarrow{b}\right), \tag{8}$$

$$\overrightarrow{l_n} = \alpha^n \overrightarrow{a} + \beta^n \overrightarrow{b}. \tag{9}$$

Proof: Compare the j-th entry on both sides of the equations. □

Recall the sum of a finite geometric series: For all real numbers $\gamma \neq 1$,

$$\sum_{j=0}^{d-1} \gamma^j = \frac{\gamma^d - 1}{\gamma - 1}. \tag{10}$$

Lemma 2.4:

$$\overrightarrow{a} \cdot \overrightarrow{a} = \begin{cases} F_d(\alpha - \beta)\alpha^{d-1} & \text{if } d \text{ is even,} \\ L_d \alpha^{d-1} & \text{if } d \text{ is odd,} \end{cases}$$

$$\overrightarrow{b} \cdot \overrightarrow{b} = \begin{cases} -F_d(\alpha - \beta)\beta^{d-1} & \text{if } d \text{ is even,} \\ L_d \beta^{d-1} & \text{if } d \text{ is odd,} \end{cases}$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 1 & \text{if } d \text{ is odd.} \end{cases}$$

Proof: By definition of dot product and Equation (10), and since $\alpha\beta = -1$,

$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{a} &= \sum_{j=0}^{d-1} \alpha^{2j} = \sum_{j=0}^{d-1} \left(\frac{-\alpha}{\beta}\right)^j = \frac{(-\alpha/\beta)^d - 1}{(-\alpha/\beta) - 1} \\ &= \frac{1}{\beta^{d-1}} \frac{(-1)^d \alpha^d - \beta^d}{-\alpha - \beta} = \alpha^{d-1}\left(\alpha^d - (-1)^d \beta^d\right). \end{aligned}$$

Suppose d is even. Then by (8)

$$\overrightarrow{a} \cdot \overrightarrow{a} = \alpha^{d-1}(\alpha - \beta)\left(\frac{\alpha^d - \beta^d}{\alpha - \beta}\right) = \alpha^{d-1}(\alpha - \beta)F_d.$$

Now suppose d is odd. Then by (9)

$$\overrightarrow{a} \cdot \overrightarrow{a} = \alpha^{d-1}\left(\alpha^d + \beta^d\right) = \alpha^{d-1} L_d.$$

The computation of $\overrightarrow{b} \cdot \overrightarrow{b}$ is similar. A direct computation using $\alpha\beta = -1$ gives $\overrightarrow{a} \cdot \overrightarrow{b}$ to be claimed. □

Consider the map which reverses each vector.

Definition 2.5: Define $\rho : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\rho(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}) = \begin{bmatrix} v_d \\ v_{d-1} \\ \vdots \\ v_1 \end{bmatrix}.$$

Observe that ρ is a vector space isomorphism. Hence, for all integers n,

$$\rho(\overrightarrow{f_n}) = \frac{1}{\alpha - \beta}\left(\alpha^n \rho(\overrightarrow{a}) - \beta^n \rho(\overrightarrow{b})\right), \tag{11}$$

$$\rho(\overrightarrow{\ell_n}) = \alpha^n \rho(\overrightarrow{a}) + \beta^n \rho(\overrightarrow{b}). \tag{12}$$

Observe that

$$\rho(\overrightarrow{a}) = \alpha^{d-1} \begin{bmatrix} 1 \\ -\beta \\ \beta^2 \\ \vdots \\ (-\beta)^{d-2} \\ (-\beta)^{d-1} \end{bmatrix} \quad \text{and} \quad \rho(\overrightarrow{b}) = \beta^{d-1} \begin{bmatrix} 1 \\ -\alpha \\ \alpha^2 \\ \vdots \\ (-\alpha)^{d-2} \\ (-\alpha)^{d-1} \end{bmatrix}.$$

Note that ρ is an isometry: For all $\overrightarrow{v}$, $\overrightarrow{w} \in \mathbb{R}^d$:

$$\rho(\overrightarrow{v}) \cdot \rho(\overrightarrow{w}) = \sum_{i=1}^{d} \rho(\overrightarrow{v})[i]\rho(\overrightarrow{w})[i] = \sum_{i=1}^{d} \overrightarrow{v}[d-i]\overrightarrow{w}[d-i] = \sum_{i=1}^{d} \overrightarrow{v}[i]\overrightarrow{w}[i] = \overrightarrow{v} \cdot \overrightarrow{w}.$$

In particular, $\rho(\overrightarrow{a}) \cdot \rho(\overrightarrow{a}) = \overrightarrow{a} \cdot \overrightarrow{a}$, and $\rho(\overrightarrow{a}) \cdot \rho(\overrightarrow{b}) = \overrightarrow{a} \cdot \overrightarrow{b}$, $\rho(\overrightarrow{b}) \cdot \rho(\overrightarrow{b}) = \overrightarrow{b} \cdot \overrightarrow{b}$ are given by Lemma 2.4. Note that ρ is an involution, so

$$\overrightarrow{v} \cdot \rho(\overrightarrow{w}) = \rho(\overrightarrow{v}) \cdot \overrightarrow{w}.$$

The following are direct results of the dot product and other definitions.

Lemma 2.6:

$$\overrightarrow{a} \cdot \rho(\overrightarrow{a}) = d\alpha^{d-1},$$

$$\overrightarrow{b} \cdot \rho(\overrightarrow{b}) = d\beta^{d-1},$$

$$\overrightarrow{a} \cdot \rho(\overrightarrow{b}) = F_d.$$

3. MAIN FIBONACCI AND LUCAS IDENTITIES

In this section we prove the shifted summation and convolution identities for the Fibonacci and Lucas numbers which were stated in the introduction. We shall only present the details for the proofs of (1) and (4) since those of (2), (3), (5), and (6) proceed similarly. We shall need the following generalizations of the Binet formulae: For all integers n and m,

$$\alpha^n\beta^m + \alpha^m\beta^n = (-1)^n L_{m-n}, \tag{13}$$

$$\alpha^n\beta^m - \alpha^m\beta^n = (-1)^{n+1}(\alpha-\beta)F_{m-n}. \tag{14}$$

Proof of Equation 1: Observe that the left side is equal to $\overrightarrow{f_n}\cdot\overrightarrow{f_m}$ by the definition of the dot product. Now expand the dot product with (8) to obtain:

$$\begin{aligned}\overrightarrow{f_n}\cdot\overrightarrow{f_m} &= \frac{1}{\alpha-\beta}\left(\alpha^n\overrightarrow{a} - \beta^n\overrightarrow{b}\right)\cdot\frac{1}{\alpha-\beta}\left(\alpha^m\overrightarrow{a} - \beta^m\overrightarrow{b}\right)\\ &= \frac{1}{5}\left(\alpha^{n+m}\overrightarrow{a}\cdot\overrightarrow{a} + \beta^{n+m}\overrightarrow{b}\cdot\overrightarrow{b} - (\alpha^n\beta^m+\alpha^m\beta^n)\overrightarrow{a}\cdot\overrightarrow{b}\right). \quad \square\end{aligned}$$

First assume d is even. Then by Lemma 2.4 and (14)

$$\overrightarrow{f_n}\cdot\overrightarrow{f_m} = \frac{F_d}{\alpha-\beta}\left(\alpha^{n+m+d-1} - \beta^{n+m+d-1}\right) = F_dF_{n+m+d-1}.$$

Now assume d is odd. Then by Lemma 2.4 and (13)

$$\begin{aligned}\overrightarrow{f_n}\cdot\overrightarrow{f_m} &= \frac{1}{5}\left(L_d(\alpha^{n+m+d-1} + \beta^{n+m+d-1}) - (\alpha^n\beta^m+\alpha^m\beta^n)\right)\\ &= \frac{1}{5}\left(L_dL_{n+m+d-1} - (-1)^nL_{m-n}\right). \quad \square\end{aligned}$$

Proof of Equation (4): Observe that the left side is equal to $\overrightarrow{f_n}\cdot\rho(\overrightarrow{f_m})$ by the definition of the dot product. Now expand the dot product with (8) and (11) and simplify using $\alpha-\beta=\sqrt{5}$, Lemma 2.6, and (13) to obtain:

$$\begin{aligned}\overrightarrow{f_n}\cdot\rho(\overrightarrow{f_m}) &= \frac{1}{\alpha-\beta}\left(\alpha^n\overrightarrow{a} - \beta^n\overrightarrow{b}\right)\cdot\frac{1}{\alpha-\beta}\left(\alpha^m\rho(\overrightarrow{a}) - \beta^m\rho(\overrightarrow{b})\right)\\ &= \frac{1}{5}\left(\alpha^{n+m}\overrightarrow{a}\cdot\rho(\overrightarrow{a}) + \beta^{n+m}\overrightarrow{b}\cdot\rho(\overrightarrow{b}) - (\alpha^n\beta^m+\alpha^m\beta^n)\overrightarrow{a}\cdot\rho(\overrightarrow{b})\right)\\ &= \frac{1}{5}\left(d\alpha^{n+m+d-1} + d\beta^{n+m+d-1} - (-1)^nL_{m-n}F_d\right)\\ &= \frac{1}{5}\left(dL_{n+m+d-1} - (-1)^nL_{m-n}F_d\right). \quad \square\end{aligned}$$

4. FURTHER FIBONACCI AND LUCAS IDENTITIES

We derive a few samples of consequences of Equations (1)–(6). Although many of these results are known or consequences of known identities, Equations (1)–(6) provide very simple proofs. The first corollary provides straightforward generalizations of Lucas' classic result $\sum_{j=1}^{n} F_j^2 = F_n F_{n+1}$. The second and third corollaries, respectively, specify the sum and difference of two squares which may not be consecutive, and the sum and difference of any twwo Fibonacci numbers with even subscripts.

Corollary 4.1: *For all integers n,*

$$\sum_{j=0}^{d-1} F_{n+j}^2 = \begin{cases} F_d F_{2n+d-1} & \text{if } d \text{ is even,} \\ \frac{1}{5}\left(L_d L_{2n+d-1} - 2(-1)^n\right) & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1} L_{n+j}^2 = \begin{cases} 5F_d F_{2n+d-1} & \text{if } d \text{ is even,} \\ L_d L_{2n+d-1} + 2(-1)^n & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1} F_{n+j} L_{n+j} = \begin{cases} F_d L_{2n+d-1} & \text{if } d \text{ is even,} \\ L_d F_{2n+d-1} & \text{if } d \text{ is odd.} \end{cases}$$

Proof: Take $m = n$ in Eq. (1), (2), and (3), respectively. □

Corollary 4.2: *For all integers m and n,*

$$F_n^2 + F_m^2 = \begin{cases} F_{n-m} F_{n+m} & \text{if } n-m \text{ is odd,} \\ \frac{1}{5}\left(L_{n-m} L_{n+m} - 4(-1)^m\right) & \text{if } n-m \text{ is even,} \end{cases}$$

$$L_n^2 + L_m^2 = \begin{cases} 5F_{n-m} F_{n+m} & \text{if } n-m \text{ is odd,} \\ L_{n-m} L_{n+m} + 4(-1)^m & \text{if } n-m \text{ is even,} \end{cases}$$

$$F_{2n} + F_{2m} = F_n L_n + F_m L_m = \begin{cases} F_{n-m} L_{n+m} & \text{if } n-m \text{ is odd,} \\ L_{n-m} F_{n+m} & \text{if } n-m \text{ is even.} \end{cases}$$

Proof: Subtract each of the equations of Corollary 4.1, with d replaced by $d-2$ and n replaced by $n+1$, from the same equation with no changes, and simplify. Write $m = n + d - 1$. □

Corollary 4.3: *For all integers m and n,*

$$F_n^2 - F_m^2 = \begin{cases} F_{n-m} F_{n+m} & \text{if } n-m \text{ is even,} \\ \frac{1}{5}\left(L_{n-m} L_{n+m} + 4(-1)^m\right) & \text{if } n-m \text{ is odd,} \end{cases}$$

$$L_n^2 - L_m^2 = \begin{cases} 5F_{n-m} F_{n+m} & \text{if } n-m \text{ is even,} \\ L_{n-m} L_{n+m} - 4(-1)^m & \text{if } n-m \text{ is odd,} \end{cases}$$

$$F_{2n} - F_{2m} = F_n L_n - F_m L_m = \begin{cases} F_{n-m} L_{n+m} & \text{if } n-m \text{ is even,} \\ L_{n-m} F_{n+m} & \text{if } n-m \text{ is odd.} \end{cases}$$

Proof: Subtract each of the equations of Corollary 4.1 from the same equation with n replaced by $n+1$, and simplify. Write $m = n + d$. □

Corollary 4.4: *For all integers ℓ, m, and n,*

$$F_{n+\ell-2}F_{n-m+\ell-1}+F_{n-1}F_{n-m}=\begin{cases}F_{\ell-1}F_{2n-m+\ell-2} & \text{if } \ell \text{ is even,}\\ \frac{1}{5}\left(L_{\ell-1}L_{2n-m+\ell-2}-2(-1)^{n-m}L_{m-1}\right) & \text{if } \ell \text{ is odd,}\end{cases}$$

$$L_{n+\ell-2}L_{n-m+\ell-1}+L_{n-1}L_{n-m}=\begin{cases}5F_{\ell-1}F_{2n-m+\ell-2} & \text{if } \ell \text{ is even,}\\ L_{\ell-1}L_{2n-m+\ell-2}+2(-1)^{n-m}L_{m-1} & \text{if } \ell \text{ is odd,}\end{cases}$$

$$F_{n+\ell-2}L_{n-m+\ell-1}+F_{n-1}L_{n-m}=\begin{cases}F_{\ell-1}L_{2n-m+\ell-2} & \text{if } \ell \text{ is even,}\\ L_{\ell-1}F_{2n-m+\ell-2}+2(-1)^{n-m}F_{m-1} & \text{if } \ell \text{ is odd.}\end{cases}$$

Proof: To prove the first equation for postive ℓ, simplify the result of subtracting (1) with d, n, and m replaced by ℓ, $n-1$ and $n-m$, respectively, from (1) with d and m replaced by $\ell-2$ and $n-m+1$ respectively. For $l \le 0$, the result follows from the $l \ge 0$ case and the identities $F_{-n}=(-1)^{n+1}F_n$ and $L_{-n}=(-1)^nL_n$. The other formulae are derived similarly. □

Corollary 4.5: *For all integers m and n,*

$$\frac{F_{n-2m}+F_{n+2m-2}}{F_{2m-1}}=L_{n-1},$$

$$\frac{F_{n+2m-1}+F_{n-2m-1}}{L_{2m}}=F_{n-1},$$

$$\frac{L_{n-2m}+L_{n+2m-2}}{F_{2m-1}}=F_{n-1},$$

$$\frac{L_{n-2m-1}+L_{n+2m-1}}{L_{2m}}=L_{n-1}.$$

In particular, for fixed n, each of the quantities on the the left-hand sides of these equations is integral and independent of m.

Proof: To prove the first equation, let $\ell=m$ be even in the first equation of Corollary 4.4, yielding $F_{n+m-2}F_{n-1}+F_{n-1}F_{n-m}=F_{m-1}F_{2n-2}$, so $(F_{n+m-2}+F_{n-m})/F_{m-1}=F_{2n-2}/F_{n-1}=F_{2(n-1)}/F_{n-1}=L_{n-1}$. Replacing m with $2m$, we have the stated conclusion.

To prove the second equation, let $\ell=m$ be odd in the first equation of Corollary 4.4, yielding $F_{n+m-2}F_{n-1}+F_{n-1}F_{n-m}=(L_{m-1}L_{2n-2}-2(-1)^{n-d}L_{m-1})/5$. Simplifying, we find that $(F_{n+m-2}+F_{n-m})/L_{m-1}=(L_{2n-2}+2(-1)^n)/(5F_{n-1})$. Now the left-hand side must be independent of m: Setting $m=1$ reveals the constant value to be F_{n-1}. Replacing m with $2m+1$ gives the result. Similar arguments based upon the second equation of Corollary 4.4 give the last two results. □

Corollary 4.6: *For all integers n,*

$$\sum_{j=0}^{d-1}(-1)^j F_{n-j}F_{n+j} = \begin{cases} (-1)^{n+1}F_dF_{d-1} & \text{if } d \text{ is even,} \\ \frac{1}{5}\left((-1)^{n+1}L_dL_{d-1}+L_{2n}\right) & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1}(-1)^j L_{n-j}L_{n+j} = \begin{cases} 5(-1)^{n}F_dF_{d-1} & \text{if } d \text{ is even,} \\ (-1)^{n}L_dL_{d-1}-L_{2n} & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1}(-1)^j F_{n+j}L_{n-j} = \begin{cases} (-1)^{n}F_{d-1}L_d-2(-1)^n & \text{if } d \text{ is even,} \\ (-1)^{n}F_{d-1}L_d+F_{2n} & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1}(-1)^j F_{n-j}L_{n+j} = \begin{cases} (-1)^{n+1}F_dL_{d-1} & \text{if } d \text{ is even,} \\ (-1)^{n+1}L_dF_{d-1}+F_{2n} & \text{if } d \text{ is odd.} \end{cases}$$

Proof: Set $m = -n$ in Eqs. (1)–(3) and simplify with $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$. □

We view the equations of Corollary 4.6 as generalizations of Cassini's identity $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$, which we recover by taking $d = 2$ in the first equation. When $d = 3$ and $d = 4$ the generalizations are $F_n^2 - F_{n-1}F_{n+1} + F_{n-2}F_{n+2} = ((-1)^{n+1}12 + L_{2n})/5$ and $F_n^2 - F_{n-1}F_{n+1} + F_{n-2}F_{n+2} - F_{n-3}F_{n+3} = 6(-1)^{n+1}$.

Corollary 4.7: *For all integers t,*

$$\sum_{j=0}^{d-1}(-1)^j F_{j+1}F_{t-j} = \begin{cases} F_dF_{t-d} & \text{if } d \text{ is even,} \\ \frac{1}{5}\left(L_dL_{t-d}+L_{t+1}\right) & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1}(-1)^j L_{j+1}L_{t-j} = \begin{cases} -5F_dF_{t-d} & \text{if } d \text{ is even,} \\ -L_dL_{t-d}+L_{t+1} & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1}(-)^j F_{j+1}L_{t-j} = \begin{cases} F_dL_{t-d} & \text{if } d \text{ is even,} \\ L_dF_{t-d}+F_{t+1} & \text{if } d \text{ is odd,} \end{cases}$$

$$\sum_{j=0}^{d-1}(-1)^j F_{t-j}L_{j+1} = \begin{cases} -F_dL_{t-d} & \text{if } d \text{ is even,} \\ -L_dF_{t-d}+F_{t+1} & \text{if } d \text{ is odd.} \end{cases}$$

Proof: Take $n = 1$ and $m = -t$ in Eqs. (1)–(3) and simplify with $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$. The last equation is derived by taking $n = -t$ and $m = 1$ in Eq. (3). □

We now give some sample consequences of (4)–(6).

Corollary 4.8: *For all integers ℓ, m, and n,*

$$\begin{aligned} F_nF_{\ell+m-1} + F_{\ell+n-1}F_m &= 2L_{n+m+\ell-1} - (-1)^n L_{m-n}L_{\ell-1}, \\ L_nL_{\ell+m-1} + L_{\ell+n-1}L_m &= 2L_{n+m+\ell-1} + (-1)^{n-m} L_{m-n}L_{\ell-1}, \\ F_nL_{\ell+m-1} + F_{\ell+n-1}L_m &= 2F_{n+m+\ell-1} - (-1)^n F_{m-n}L_{\ell-1}. \end{aligned}$$

Proof: The proof is similar to that of Corollary 4.4. Here the equations $F_{n-1}+F_{n+1}=L_n$ and $L_{n-1}+L_{n+1}=5F_n$ for all integers n (see [1,2]) are useful in the simplification. □

5. LINEAR ALGEBRA AND FUTURE DIRECTIONS

This paper is based upon part of the Master's thesis of the second author (Salter) [4] performed under the joint supervision of the other two authors. This thesis dealt with the linear algebra of the Fibonacci and Lucas vectors. Observe that we have in fact proven the following dot product formulae.

Theorem 5.1: *For all integers m and n,*

$$\overrightarrow{f_n}\cdot\overrightarrow{f_m}=\begin{cases} F_dF_{n+m+d-1} & \text{if } d \text{ is even,}\\ \frac{1}{5}\left(L_dL_{n+m+d-1}-(-1)^nL_{m-n}\right) & \text{if } d \text{ is odd,}\end{cases}$$

$$\overrightarrow{\ell_n}\cdot\overrightarrow{\ell_m}=\begin{cases} 5F_dF_{n+m+d-1} & \text{if } d \text{ is even,}\\ L_dL_{n+m+d-1}+(-1)^nL_{m-n} & \text{if } d \text{ is odd,}\end{cases}$$

$$\overrightarrow{f_n}\cdot\overrightarrow{\ell_m}=\begin{cases} F_dL_{n+m+d-1} & \text{if } d \text{ is even,}\\ L_dF_{n+m+d-1}+(-1)^{n+1}F_{m-n} & \text{if } d \text{ is odd,}\end{cases}$$

$$\overrightarrow{f_n}\cdot\rho(\overrightarrow{f_m})=\frac{1}{5}\left(dL_{n+m+d-1}-(-1)^nL_{m-n}F_d\right),$$

$$\overrightarrow{\ell_n}\cdot\rho(\overrightarrow{\ell_m})=dL_{n+m+d-1}+(-1)^nL_{m-n}F_d,$$

$$\overrightarrow{f_n}\cdot\rho(\overrightarrow{\ell_m})=dF_{n+m+d-1}+(-1)^{n+1}F_{m-n}F_d.$$

Corollary 5.2: *For all integers n,*

$$\|\ \overrightarrow{f_n}^2\ \| =\begin{cases} F_dF_{2n+d-1} & \text{if } d \text{ is even,}\\ \frac{1}{5}\left(L_dL_{2n+d-1}-2(-1)^n\right) & \text{if } d \text{ is odd,}\end{cases}$$

$$\|\ \overrightarrow{\ell_n}^2\ \| =\begin{cases} 5F_dF_{2n+d-1} & \text{if } d \text{ is even,}\\ L_dL_{2n+d-1}+2(-1)^n & \text{if } d \text{ is odd.}\end{cases}$$

Corollary 5.3: *For all integers m and n,*

$$\overrightarrow{\ell_n}\cdot\overrightarrow{\ell_m}=\begin{cases} 5\overrightarrow{f_n}\cdot\overrightarrow{f_m} & \text{if } d \text{ is even,}\\ 5\overrightarrow{f_n}\cdot\overrightarrow{f_m}+2(-1)^nL_{m-n} & \text{if } d \text{ is odd.}\end{cases}$$

By Theorem 2.3, all Fibonacci and Lucas vectors of a given length lie in a plane. In future work we shall further develop the linear algebraic and geometric results concerning the Fibonacci and Lucas vectors studied in [4].

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