VIETA-LIKE PRODUCTS OF NESTED RADICALS WITH FIBONACCI AND LUCAS NUMBERS

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ABSTRACT

We present two infinite products of nested radicals involving Fibonacci and Lucas numbers. These products resemble Vieta's classical product of nested radicals for $2/\pi$. A modern derivation of Vieta's product involves trigonometric functions, while our product involves similar manipulations involving hyperbolic functions.

The beautiful infinite product of radicals

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots \tag{1}$$

due to Vieta [1] in 1592, is one of the oldest noniterative analytical expressions for π. It is the purpose of this note to prove the following two Vieta-like products

$$\frac{\sqrt{5}F_N}{2N\log\phi}=\sqrt{\frac{1}{2}+\frac{L_n}{4}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{L_n}{4}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{L_N}{4}}}}\cdots \tag{2}$$

for N even, and

$$\frac{L_N}{2N\log\phi}=\sqrt{\frac{1}{2}+\frac{\sqrt{5}F_N}{4}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{5}F_N}{4}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{5}F_N}{4}}}}\cdots \tag{3}$$

for N odd. Here N is a positive integer, F_N and L_N are the Fibonacci and Lucas numbers, and $\phi=\frac{1+\sqrt{5}}{2}$ is the golden section.

First we must explore a few exact values of the hyperbolic functions. Notice that $\frac{2}{\sqrt{5}}\sinh(N\log\phi)=\frac{1}{\sqrt{5}}\left(e^{N\log\phi}-e^{-N\log\phi}\right)=\frac{1}{\sqrt{5}}\left(\phi^N-\left(\frac{1}{\phi}\right)^N\right)=F_N$ for even N. (This last equality follows from Binet's formula [2], $F_n=\frac{1}{\sqrt{5}}\left(\phi^n-\left(-\frac{1}{\phi}\right)^n\right)$, true for all positive n.) For odd N we have $2\sinh(N\log\phi)=e^{N\log\phi}-e^{-N\log\phi}=\phi^N-\left(\frac{1}{\phi}\right)^N=L_N$. (This last equality follows from the Binet-like formula $L_n=\phi^n+\left(-\frac{1}{\phi}\right)^n$, which is true for all positive n.) Thus we have derived

$$\sinh(N\log\phi)=\begin{cases}\frac{\sqrt{5}}{2}F_N & \text{for } N \text{ even}\\ \frac{1}{2}L_N & \text{for } N \text{ odd}\end{cases}. \tag{4}$$

In a similar way we can derive

$$\cosh(N\log\phi) = \begin{cases} \frac{1}{2}L_N & \text{for } N \text{ even} \\ \frac{\sqrt{5}}{2}F_N & \text{for } N \text{ odd} \end{cases}. \tag{5}$$

Notice that in some ways the number $\log\phi$ acts with the hyperbolic functions as π does with the trigonometric functions. The hyperbolic functions of certain rational multiples of $\log\phi$ can be expressed as exact values.

To derive (2) and (3) we start by applying the double angle formula for the hyperbolic sine function p times to obtain

$$\begin{aligned} \sinh x &= 2\cosh\frac{x}{2}\sinh\frac{x}{2} \\ &= 2^2\cosh\frac{x}{2}\cosh\frac{x}{2^2}\sinh\frac{x}{2^2} \\ &= 2^3\cosh\frac{x}{2}\cosh\frac{x}{2^2}\cosh\frac{x}{2^3}\sinh\frac{x}{2^3} \\ &\dots \end{aligned}$$

$$\sinh x = 2^p\cosh\frac{x}{2}\cosh\frac{x}{2^2}\cosh\frac{x}{2^3}\cdots\cosh\frac{x}{2^p}\sinh\frac{x}{2^p}. \tag{6}$$

We evaluate each of the hyperbolic cosine factors in (6) in terms of $\cosh x$ by repeated use of the half-angle formula for the hyperbolic cosine.

$$\begin{aligned} \cosh\frac{x}{2} &= \sqrt{\frac{1}{2}+\frac{1}{2}\cosh x} \\ \cosh\frac{x}{2^2} &= \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}} \\ &\dots \\ \cosh\frac{x}{2^p} &= \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}}}}_{(p \text{ radicals})}. \end{aligned} \tag{7}$$

Combining (7) with (6) and dividing by x we obtain

$$\frac{\sinh x}{x} = \frac{2^p}{x}\sinh\left(\frac{x}{2^p}\right)\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}}}.$$

If we let p tend to infinity we get (since $\lim_{a\to 0}(\sinh\alpha)/\alpha = 1$),

$$\frac{\sinh x}{x} = \prod_{n=1}^{\infty}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}}}.$$

$$(n \text{ radicals}) \tag{8}$$

Now let $x = N\log\phi$ in (8) and use (4) and (5) to obtain at once our desired products (2) and (3). This completes our proof.

It is interesting to notice that a common derivation of the original Vieta product (1) proceeds like our derivation of (8) with hyperbolic functions of x replaced by trigonometric functions of θ. In the final step where we set $x = N\log\phi$ in the hyperbolic functions to obtain (2) and (3), one sets $\theta = \pi/2$ in the trigonometric functions to obtain (1).

This note was motivated by a discussion with Richard Askey in which he showed how the Fibonacci and Lucas numbers are related to the hyperbolic functions.

REFERENCES

[1] L. Berggren, J. Borwein and P. Borwein. *Pi, A Source Book*, Springer, New York, 1997, pp. 53-67.

[2] N. N. Vorob'ev. *Fibonacci Numbers*, Pergamon Press, 1961, pp. 20-28.

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