CATALAN NUMBERS, FACTORIALS, AND SUMS OF ALIQUOT PARTS

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ABSTRACT

Let $\mathcal{A}$ be the set of all Catalan numbers and factorials. In this note, we look at positive integers $n \in \mathcal{A}$ whose sum of aliquot parts also belongs to $\mathcal{A}$.

1. INTRODUCTION

For a positive integer n we write $\sigma(n)$ for the sum of all the positive integer divisors of n and $s(n) = \sigma(n) - n$ for the sum of proper divisors of n. We recall that $s(n)$ is sometimes referred to as the sum of *aliquot parts* of n. A number n is called perfect if $s(n) = n$. If n is not perfect but $s(s(n)) = n$, then the pair $(n, s(n))$ is called *amicable.* More generally, an *aliquot cycle* of length k is a cycle of k positive integers $(n_1, n_2, \ldots, n_k)$ such that if we set $n_{k+1} := n_1$ then $n_i = s(n_{i-1})$ holds for all $i = 2, \ldots, k+1$. It is conjectured that any positive integer n belongs to some aliquot cycle of length k for some positive integer k.

In this paper, we fix certain infinite subsets of positive integers, say $\mathcal{A}$ and $\mathcal{B}$ and we try to determine all $n \in \mathcal{A}$ such that $s(n) \in \mathcal{B}$. Our sets $\mathcal{A}$ and $\mathcal{B}$ will be the subsets of all Catalan numbers or factorials. Recall that a Catalan number is a number of the form $C_n = \frac{1}{n+1}\binom{2n}{n}$ for integer $n \geq 0$. Finally, a factorial is simply a positive integer of the form $n!$ for some integer $n \geq 0$.

We record our results as follows.

Theorem 1: *The only solutions in positive integers* (n, m) *for the equation*

$$s(C_n) = m! \tag{1}$$

are the trivial solutions $(2, 1)$ *and* $(3, 1)$.

Theorem 2: *The only solution in positive integers* (m, n) *for the equation*

$$s(m!) = C_n \tag{2}$$

is the trivial solution $(2, 1)$.

Theorem 3: *The only solutions in positive integers* (n, m) *for the equation*

$$s(n!) = m! \tag{3}$$

are the trivial solutions $(2,1)$ *and* $(3,3)$.

Theorem 4: *The only solutions in positive integers* (n,m) *for the equation*

$$s(C_n) = C_m \tag{4}$$

are the trivial solutions $(2,1)$ *and* $(3,1)$.

Throughout this paper, for a positive integer k we write $v_2(k) = \alpha$ if $2^\alpha \| k$. We refer to $v_2(k)$ as the 2-*valuation of* k. We also let $\ell_2(k)$ denote the sum of the binary digits of k. We shall use the obvious inequality

$$\ell_2(k) \le \frac{\log k}{\log 2} + 1$$

as well as the known fact that

$$v_2(k!) = k - \ell_2(k).$$

We finally let $\pi(k)$ denote the number of primes $p \le k$.

2. PROOF OF THEOREM 1

First we compare the 2-valuation of both sides of (1). Since

$$\begin{aligned} v_2\binom{2n}{n} &= v_2((2n)!) - 2v_2(n!) \\ &= 2n - \ell_2(2n) - 2n + 2\ell_2(n) \\ &= \ell_2(n) \le \frac{\log n}{\log 2} + 1, \end{aligned}$$

we have

$$v_2(C_n) = v_2\binom{2n}{n} - v_2(n+1) \le v_2\binom{2n}{n} \le \frac{\log n}{\log 2} + 1. \tag{5}$$

Since C_n is divisible exactly once by all primes p such that $n+1 < p \le 2n$, we have

$$v_2(\sigma(C_n)) \ge \sum_{n+1<p\le 2n} v_2(p+1) \ge \pi(2n) - \pi(n+1).$$

Since

$$\pi(2n) - \pi(n+1) \ge \frac{n}{2\log n} \tag{6}$$

for all $n \geq 7$ (see Rosser and Schoenfeld [1]), we have

$$v_2(\sigma(C_n)) \geq \frac{n}{2\log n}, \qquad \text{whenever } n \geq 7. \tag{7}$$

For $n \geq 54$ we also have

$$\frac{n}{2\log n} > \frac{\log n}{\log 2} + 1.$$

Thus, by (5) and (7), we have $v_2(\sigma(C_n)) > v_2(C_n)$, and so again if $n \geq 54$ we get

$$v_2\left(\sigma(C_n) - C_n\right) = v_2(C_n) \leq \frac{\log n}{\log 2} + 1. \tag{8}$$

We also have

$$v_2(m!) = m - \ell_2(m) \geq m - \frac{\log m}{\log 2} - 1. \tag{9}$$

Next, we obtain a lower bound for the left-hand side of (1). Since C_n is divisible exactly once by all primes p such that $n+1 < p \leq 2n$, we have

$$\sigma(C_n) \geq C_n \prod_{n+1<p\leq 2n} \left(1 + \frac{1}{p}\right) \geq C_n \left(1 + \frac{1}{2n}\right)^{\pi(2n)-\pi(n+1)},$$

and so, by estimate (6), we have

$$\sigma(C_n) \geq C_n \left(1 + \frac{1}{2n}\right)^{\frac{n}{2\log n}}, \qquad \text{whenever } n \geq 7.$$

Taking logarithms in the last inequality above we get

$$\begin{aligned}
\log(\sigma(C_n)) &\geq \log C_n + \frac{n}{2\log n} \log\left(1 + \frac{1}{2n}\right) \\
&\geq \log C_n + \frac{n}{2\log n}\left(\frac{1}{2n} - \frac{1}{8n^2}\right) \\
&= \log C_n + \frac{1}{2\log n}\left(\frac{1}{2} - \frac{1}{8n}\right);
\end{aligned}$$

equivalently,

$$\sigma(C_n) \ge C_n \cdot \exp\left(\frac{1}{4\log n} - \frac{1}{16n\log n}\right). \tag{10}$$

Recalling the known inequality

$$C_n = \frac{1}{n+1}\binom{2n}{n} \ge \frac{2^{2n}}{(n+1)^2},$$

we get

$$\begin{aligned}\sigma(C_n) - C_n &\ge C_n \cdot \left(\exp\left(\frac{1}{4\log n} - \frac{1}{16n\log n}\right) - 1\right)\\ &\ge \frac{2^{2n}}{(n+1)^2} \cdot \left(\exp\left(\frac{1}{4\log n} - \frac{1}{16n\log n}\right) - 1\right)\\ &\ge n^{2\log n \log\log n}.\end{aligned}$$

The last inequality claimed above holds for all $n \ge 28$. We have thus shown that if $n \ge 54$ then

$$\sigma(C_n) - C_n \ge n^{2\log n \log\log n}. \tag{11}$$

In particular, $m! > n^{2\log n \log\log n}$, which for $n \ge 54$ implies that $m \ge 10$. By (1), (8) and (9), we have, for $n \ge 54$,

$$\frac{\log n}{\log 2} + 1 \ge m - \frac{\log m}{\log 2} - 1,$$

which implies $n \ge 2^{m-2}/m$. Since $2^{m-2}/m > e^{\sqrt{m}}$ for $m \ge 10$, we have

$$n \ge e^{\sqrt{m}}, \qquad \text{whenever} \quad m \ge 10 \quad \text{and} \quad n \ge 54. \tag{12}$$

Finally, by (11) and (12), we get that if $n \ge 54$, then

$$\begin{aligned}\sigma(C_n) - C_n &\ge n^{2\log n \log\log n}\\ &\ge (e^{\sqrt{m}})^{\sqrt{m}\log m}\\ &= m^m > m!,\end{aligned}$$

which contradicts (1). Thus, any solutions to (1) must be in the range $n < 54$. Computation then reveals that the only such solutions are $(n, m) = (2, 1)$ and $(3, 1)$. □

3. PROOF OF THEOREM 2

Recalling (9), we have

$$v_2(m!) \ge m - \frac{\log m}{\log 2} - 1.$$

Since $m!$ is divided exactly once by all primes p such that $m/2 < p \le m$, we have

$$v_2(\sigma(m!)) \ge \sum_{\frac{m}{2} < p \le m} v_2(p+1) \ge \pi(m) - \pi(m/2)$$

$$\ge \frac{m}{3\log m}$$

(again, see Rosser and Schoenfeld [1]) for $m \ge 18$. Since

$$m - \frac{\log m}{\log 2} - 1 > \frac{m}{3\log m}$$

for $m \ge 4$, we have that for all $m \ge 18$,

$$v_2(\sigma(m!) - m!) \ge \frac{m}{3\log m}. \tag{13}$$

On the other hand, recalling (5), we also have

$$v_2(C_n) \le \frac{\log n}{\log 2} + 1.$$

Therefore (13) and (5) together imply that for $m \ge 18$ we have

$$\frac{\log n}{\log 2} + 1 \ge \frac{m}{3\log m}.$$

Note that for all $m \ge 225$ we also have that $m > 3\log m(3 + 2\log m)$, which in turn implies that

$$\frac{m}{3\log m} > 3 + 2\log m.$$

Thus, for $m \geq 225$, we have

$$\frac{\log n}{\log 2} + 1 > 3 + 2\log m.$$

The above inequality implies that $\log n > 2 + 2\log m > 4m\log m$, which in turn leads to $n\log 2 > 2m\log m$, or,

$$2^n > m^{2m}.$$

Since $m \geq 225$, the last inequality above certainly implies that $n \geq 7$. But for $n \geq 7$ we also have

$$C_n > \frac{2^{2n}}{(n+1)(2n+1)} > 2^n,$$

and so

$$s(m!) \leq \sum_{k=1}^{m!-1} k = \frac{m!(m!-1)}{2} < \frac{m^m(m^m-1)}{2} < m^{2m}.$$

Thus, we get the contradiction $C_n > 2^n > m^{2m} > \sigma(m!) - m!$ if $m \geq 225$. Computation now shows that the only solution to (2) in the remaining range $m \leq 224$ is $(n, m) = (2, 1)$. □

4. PROOF OF THEOREM 3

We shall assume (3) holds for $n \geq 4$; it is easy to see that the only solutions when $n \leq 3$ are those stated in Theorem 3. Thus $12 \mid n!$, and so $n!$ is abundant; this implies $s(n!) > n!$, and so by (3)

$$m! > n!. \tag{14}$$

Next, we note that

$$\sigma(n!) = n!\sum_{d|n!}\frac{1}{d} < n!\sum_{k=1}^{n!}\frac{1}{k} < n!(1+\log n!) < n!(1+n\log n).$$

Thus we get $s(n!) < n! \cdot n\log n < n! \cdot n^2 < (n+2)!$. Thus by (3) and (14),

$$n! < m! < (n+2)!,$$

which implies $n < m < n+2$. Hence, we have $m = n+1$. Thus, (3) becomes $s(n!) = (n+1)!$, or, equivalently, $\sigma(n!) = n!(n+2)$. We may state this as

$$\frac{\sigma(n!)}{n!} = n+2. \tag{15}$$

The function $\sigma(n)/n$ is multiplicative and for prime p and $a \geq 1$ we have

$$\frac{\sigma(p^a)}{p^a} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^a} < \sum_{k=0}^{\infty} \frac{1}{p^k} = \frac{p}{p-1}.$$

Therefore

$$\frac{\sigma(n!)}{n!} < \prod_{p \leq n} \frac{p}{p-1} < e^{\gamma} \sum_{k=1}^{n} \frac{1}{k},$$

the right hand inequality following for all $n \geq 1$ by equation (3.31) in Rosser and Schoenfeld [1]—note here that γ denotes Euler's gamma constant. As

$$\sum_{k=1}^{n} \frac{1}{k} < 1 + \log n,$$

we get by (15),

$$n + 2 < e^{\gamma}(1 + \log n),$$

but this statement is clearly false when $n \geq 4$, which we assumed. Therefore the only solutions to (3) are $(m, n) = (2, 1)$ and $(3, 3)$. □

5. PROOF OF THEOREM 4

In (4), we shall assume that $n = m \pm t$ for some nonnegative integer t. Our immediate goal is to obtain a bound on t. It is easy to see that $C_{m+1}/C_m \geq 3$ for all m. In fact,

$$\frac{C_{m+1}}{C_m} = \frac{4m+2}{m+2} \in [3, 4), \qquad \text{whenever } m \geq 4. \tag{16}$$

We now consider two cases separately, namely when $m \geq n$ and when $m < n$, respectively.

If $m \geq n$, then by (16), we have $C_m = C_{n+t} \geq 3^t C_n$. Furthermore, since

$$\sigma(C_n) < C_n \sum_{k=1}^{2^{2n}} \frac{1}{k} < C_n(1 + 2n \log 2),$$

we have

$$3^t C_n \leq C_m = s(C_n) < C_n(2n \log 2) < 2nC_n,$$

which implies

$$t < \frac{\log 2n}{\log 3}. \tag{17}$$

Assume now that $m < n$. Recall that, by estimate (10), we have that

$$C_m = s(C_n) \geq C_n \left(\exp\left(\frac{1}{4\log n} - \frac{1}{16n\log n} \right) - 1 \right) > \frac{3C_n}{16\log n},$$

whenever $n \geq 7$, where in the rightmost inequality above we used the fact that $e^x - 1 > x$ holds for all positive numbers x. Thus, by containment (4), we get

$$\frac{C_n}{4^t} \geq C_m = s(C_n) > \frac{3C_n}{16\log n},$$

and so $3^t < 4^t < (16\log n)/3 < 2n$, where the last inequality holds for all $n \geq 2$. This gives us again that $t < (\log 2n)/(\log 3)$. We have thus shown that $|m - n| < (\log 2n)/(\log 3)$. We let $T = (\log 2n)/(\log 3)$ and denote by $\mathcal{I}$ the interval $\mathcal{I} = (n + 1 + T, 2n - 2T]$. Since $\mathcal{I} \subset (n + 1, 2n] \cap (m + 1, 2m]$, we have that $p \mid C_n$ and $p \mid C_m$ for all primes $p \in \mathcal{I}$. Thus, by equation (4), we have that $p \mid \sigma(C_n)$ as well for all primes $p \in \mathcal{I}$. Since $p \mid C_n$ for all primes $p \in \mathcal{I}$, we have

$$\prod_{p\in\mathcal{I}} (p + 1) \mid \sigma(C_n).$$

Since the largest prime factor of the number appearing in the left hand side of the last divisibility relation above is $\leq (2n - 2T + 1)/2 \leq n$ (because all such primes p are odd), we get that the number appearing in the left hand side of the above divisibility relation does not have any prime factor $p \in \mathcal{I}$. We now conclude that in fact

$$\prod_{p\in\mathcal{I}} p(p + 1) \mid \sigma(C_n).$$

Thus,

$$\sigma(C_n) \geq \prod_{p\in\mathcal{I}} p(p + 1) > n^{2(\pi(2n-2T)-\pi(n+T))}. \tag{18}$$

We now recall from [1] that

$$\pi(x) > \frac{x}{\log x - 0.5}, \qquad \text{whenever } x \geq 67, \tag{19}$$

and

$$\pi(x) < \frac{x}{\log x - 1.5}, \qquad \text{whenever } x \geq e^{2/3}. \tag{20}$$

Using these inequalities, we checked that

$$\pi(2n - 2T) - \pi(n + T) > \frac{7n}{10 \log n}, \qquad \text{whenever } n \geq 117. \tag{21}$$

To check (21), note that by inequalites (19) and (20) we have

$$\pi(2n - 2T) > \frac{2n - 2T}{\log(2n - 2T) - 0.5}, \qquad \text{whenever } n > 67$$

(note that $2n - 2T > n$ when $n > 67$, because this inequality is implied by $n > 2\log(2n)$, or $e^n > 4n^2$, and this is certainly true for $n > 67$), and

$$\pi(n + T) < \frac{n + T}{\log(n + T) - 1.5}, \qquad \text{whenever } n > e^{3/2}.$$

Hence, in order to prove that inequality (21) holds, it suffices to check that

$$\frac{2n - 2T}{\log(2n - 2T) - 0.5} - \frac{n + T}{\log(n + T) - 1.5} > \frac{7n}{10 \log n} \tag{22}$$

holds for all $n \geq 117$ with $T = (\log 2n)/(\log 3)$. We checked with *Mathematica* that inequality (22) holds for all $n > 2224$, and we then checked that inequality (21) holds for all positive integers $n \in [117, 2224]$, which completes the proof of inequality (21). Inequality (18) in conjunction with inequality (21) gives us that

$$\sigma(C_n) > n^{\frac{7n}{5 \log n}}, \qquad \text{whenever } n \geq 117.$$

On the other hand, we also have

$$\sigma(C_n) < C_n(1 + 2n\log 2) < \frac{2^{2n}}{(n+1)^2}(1 + 2n\log 2),$$

and the last two inequalities above imply that

$$2^{2n}(1 + 2n\log 2) > n^2 e^{\frac{7}{5}n},$$

which in turn leads to

$$2^{2n+1} > ne^{\frac{7}{5}n}.$$

Taking logarithms, we get

$$2n\log 2 + \log 2 > \log n + \frac{7}{5}n,$$

which in turn leads to $2\log 2 > 7/5$, which is false. In conclusion, if (4) has any solutions at all, then they must occur only when $n < 117$. Computation then shows that when $n < 117$, the equation $s(C_n) = C_m$ is satisfied only for the pairs $(n, m) = (2, 1)$ or $(3, 1)$.

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