

DERIVATIVE POLYNOMIALS FOR TANH, TAN, SECH AND SEC IN EXPLICIT FORM

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ABSTRACT

The derivative polynomials for the hyperbolic and trigonometric tangent, cotangent and secant are found in explicit form, where the coefficients are given in terms of the Stirling numbers of the second kind. As an application we evaluate some integrals and also give the reflection formula for the Polygamma function in explicit form.

1. INTRODUCTION

The derivative polynomials for tangent and secant, correspondingly P_m and Q_m , are defined by the equations ($m = 0, 1, \dots$):

$$\left(\frac{d}{d\theta}\right)^m \tan \theta = P_m(\tan \theta), \quad (1.1)$$

$$\left(\frac{d}{d\theta}\right)^m \sec \theta = \sec \theta Q_m(\tan \theta)$$

(see Hoffman [5]). Thus, $P_m(z)$ is a polynomial of degree $m + 1$ and $Q_m(z)$ is a polynomial of degree m . The derivative polynomials have generating functions

$$\tan(\theta + x) = \sum_{m=0}^{\infty} P_m(\tan \theta) \frac{x^m}{m!},$$

$$\frac{\sec(\theta + x)}{\sec(\theta)} = \sum_{m=0}^{\infty} Q_m(\tan \theta) \frac{x^m}{m!},$$

which result from Taylor's expansions centered at $x = 0$ of the function $f(x) = \tan(\theta + x)$ and $g(x) = \sec(\theta + x)$ for a fixed θ (cf. [4, p. 287], [5], [8]). Recurrence relations are known for both P_m and Q_m , see [5]. Hoffman presented a number of properties of these polynomials in [5] and [6], but did not compute them explicitly. Detailed work on the derivative polynomials can be found in Schwatt [13].

In this note we provide an explicit form of the derivative polynomials for tanh, tan, sech and sec, where the coefficients are written in terms of Stirling numbers and $P_m(0), Q_m(0)$ are

expressed in terms of Bernoulli and Euler numbers, respectively. We also give some applications, complementing the interesting applications in [5].

It is well-known that the Eisenstein series

$$e_r(z) = \frac{1}{z^r} + \sum_{k=1}^{+\infty} \left(\frac{1}{(z+k)^r} + \frac{1}{(z-k)^r} \right)$$

(r - a positive integer) can be written as a polynomial of $e_1(z)$, see [10, Chapter 11]. In Section 5 we give this polynomial in explicit form (equation (5.3)). In that section we consider also the Polygamma functions

$$\psi_n(z) := \left(\frac{d}{dz} \right)^{n+1} \log \Gamma(z),$$

and provide an explicit form for their reflection formula:

$$\psi_n(z) - (-1)^n \psi_n(1-z) = -\pi \left(\frac{d}{dz} \right)^n \cot \pi z. \tag{1.2}$$

Finally, in Section 6 the derivative polynomials are used to evaluate some integrals.

2. PREREQUISITES

Let $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ be the Stirling numbers of the second kind; see e.g., [4, Ch 6]. Consider the geometric polynomials

$$\omega_n(x) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} k! x^k, \tag{2.1}$$

which serve as derivative polynomials for the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1,$$

according to the derivative operator xD ($D = d/dx$), that is,

$$(xD)^m \left\{ \frac{1}{1-x} \right\} = \frac{1}{1-x} \omega_m \left(\frac{x}{1-x} \right), \quad m = 0, 1, 2, \dots, \quad x \neq 1 \tag{2.2a}$$

or

$$\sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} \omega_m \left(\frac{x}{1-x} \right), \text{ when } |x| < 1, \tag{2.2b}$$

see [2], [3, (4.10)].

We now define a more general class of polynomials. For any $a, b \in \mathbb{C}$ set

$$p_m(z; a, b) := \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k \omega_k(z), \tag{2.3a}$$

or

$$p_m(z; a, b) = \sum_{j=0}^m \left[j! \sum_{k=j}^m \binom{m}{k} \begin{Bmatrix} k \\ j \end{Bmatrix} a^{m-k} b^k \right] z^j. \tag{2.3b}$$

Note that when $a = 0$:

$$p_m(z; 0, b) = b^m \omega_m(z). \tag{2.4}$$

The importance of these polynomials comes from the following lemma.

Lemma 1: For every $a, b \in \mathbb{C}$ with $x^b \neq 1$ and every $m = 0, 1, \dots$, we have:

$$(xD)^m \frac{x^a}{1-x^b} = \frac{x^a}{1-x^b} p_m \left(\frac{x^b}{1-x^b}; a, b \right), \tag{2.5a}$$

$$(xD)^m \frac{x^a}{1+x^b} = \frac{x^a}{1+x^b} p_m \left(\frac{-x^b}{1+x^b}; a, b \right), \tag{2.5b}$$

Proof: We prove (2.5a). Let $|x^b| < 1$. Then

$$\begin{aligned} (xD)^m \frac{x^a}{1-x^b} &= (xD)^m \sum_{n=0}^{\infty} x^{a+bn} = \sum_{n=0}^{\infty} (a+bn)^m x^{a+bn} \\ &= x^a \sum_{n=0}^{\infty} \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k n^k x^{bn} = x^a \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k \left\{ \sum_{n=0}^{\infty} n^k (x^b)^n \right\} \\ &= x^a \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k \left\{ \frac{1}{1-x^b} \omega_k \left(\frac{x^b}{1-x^b} \right) \right\} = \frac{x^a}{1-x^b} p_m \left(\frac{x^b}{1-x^b}; a, b \right). \end{aligned}$$

Now (2.5a) follows for every $x^b \neq 1$ by analytic continuation. (2.5b) is proved in the same manner.

3. THE DERIVATIVE POLYNOMIALS FOR COTH, TANH, TAN, AND COT

Define the derivative polynomial $C_m(z)$ of degree $m + 1$ by

$$\left(\frac{d}{d\theta}\right)^m \coth\theta = C_m(\coth\theta). \tag{3.1}$$

Therefore, $C_0(z) = 1$, $C_1(z) = 1 - z^2$, $C_2(z) = 2z^3 - 2z$, etc.

Proposition 1: For $m \geq 1$ one has

$$C_m(z) = (-2)^m(z + 1) \omega_m\left(\frac{z - 1}{2}\right) \tag{3.2}$$

where ω_m is the geometric polynomial (2.1).

Proof: We have

$$\coth\theta = \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}} = \frac{2}{1 - e^{-2\theta}} - 1.$$

Let $x = e^\theta$. Then $d/d\theta = xD$ and from Lemma 1,

$$\left(\frac{d}{d\theta}\right)^m \coth\theta = 2(xD)^m \frac{1}{1 - x^{-2}} = 2(-2)^m \frac{1}{1 - x^{-2}} \omega_m\left(\frac{x^{-2} - 1}{2}\right).$$

Set $z = \coth\theta$. Then

$$\frac{1}{1 - e^{-2\theta}} = \frac{e^\theta}{e^\theta - e^{-\theta}} = \frac{\cosh\theta + \sinh\theta}{2 \sinh\theta} = \frac{1}{2}(z + 1),$$

and,

$$\frac{e^{-2\theta}}{1 - e^{-2\theta}} = \frac{e^{-\theta}}{e^\theta - e^{-\theta}} = \frac{\cosh\theta - \sinh\theta}{2 \sinh\theta} = \frac{1}{2}(z - 1),$$

so we arrive at (3.2).

The higher derivatives of tanh are formed in the same pattern as those of coth, as both functions satisfy the same differential equation

$$y' = 1 - y^2.$$

Therefore, C_m are also the derivative polynomials of tanh,

$$\left(\frac{d}{d\theta}\right)^m \tanh\theta = C_m(\tanh\theta). \tag{3.3}$$

Explicitly, for $m \geq 1$

$$C_m(z) = (-2)^m (z + 1) \sum_{k=0}^m \frac{k!}{2^k} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (z - 1)^k. \tag{3.4}$$

Thus, $C_m(-1) = 0$, $C_m(1) = 2(-2)^m \left\{ \begin{matrix} m \\ 0 \end{matrix} \right\} = 0(m > 0)$ and

$$C_m(0) = (-2)^m \omega_m \left(\frac{-1}{2}\right) = (-2)^m \sum_{k=0}^m (-1)^k \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{k!}{2^k}. \tag{3.5}$$

The numbers $C_m(0)/m!$ are the coefficients in the Maclaurin expansion of $\tanh\theta$. In view of [4, p. 317, problem 6.76] one has

$$\sum_{k=0}^m (-1)^k \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{k!}{2^k} = \frac{2}{m + 1} (1 - 2^{m+1}) B_{m+1},$$

where B_k are the Bernoulli numbers, and then from (3.5)

$$C_m(0) = \frac{(-1)^m}{m + 1} 2^{m+1} (1 - 2^{m+1}) B_{m+1}. \tag{3.6}$$

Note that for even indices $m = 2k$ the numbers $C_{2k}(0)$ equal zero as the hyperbolic tangent is an odd function. When $m = 2k - 1$:

$$C_{2k-1}(0) = \frac{1}{2k} 2^{2k} (2^{2k} - 1) B_{2k}. \tag{3.7}$$

Proposition 2: For $m \geq 1$ the derivative polynomials for the tangent are:

$$P_m(z) = i^{m+1}2^m(1 - iz)\omega_m\left(\frac{1 + iz}{-2}\right). \tag{3.8}$$

This result follows, in the same way as in the proof of Proposition 1, from the representation

$$\tan\theta = \frac{2i}{1 + e^{2i\theta}} - i.$$

We can obtain $P_m(z)$ also from Proposition 1 by using the relation

$$\tan(\theta) = -i \tanh(i\theta).$$

Thus,

$$P_m(\tan\theta) = \left(\frac{d}{d\theta}\right)^m \tan\theta = -i^{m+1}C_m(\tanh(i\theta)) = -i^{m+1}C_m(i \tan\theta), \tag{3.9}$$

and from (3.2)

$$P_m(z) = -i^{m+1}(-2)^m(iz + 1)\omega_m\left(\frac{iz - 1}{2}\right), \tag{3.10}$$

which is the same as (3.8), as $\tan\theta$ is an odd function and from (1.1)

$$P_m(-z) = (-1)^{m+1}P_m(z).$$

In view of (2.1),

$$P_m(z) = -i^{m+1}(-2)^m(iz + 1)\sum_{k=0}^m \frac{k!}{2^k} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (iz - 1)^k, \tag{3.11}$$

and in particular,

$$P_m(0) = -i^{m+1}(-2)^m \sum_{k=0}^m (-1)^k \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{k!}{2^k} \tag{3.12}$$

(obtained independently by Knopf [7]). Here $P_{2k}(0) = 0$ and the numbers

$$P_{2k-1}(0) = (-1)^{k+1}C_{2k-1}(0) \tag{3.13}$$

(see (3.9)) are known as the *tangent numbers*. In view of (3.7) we have the well-known representation of these numbers ([4, p. 287], [11, p. 154], [10, p. 221]):

$$P_{2k-1}(0) = \frac{(-1)^{k+1}}{2k} 2^{2k} (2^{2k} - 1) B_{2k}. \quad (3.14)$$

Finally, the derivative polynomials for the cotangent are

$$-P_m(-z) = (-1)^m P_m(z). \quad (3.15)$$

This follows immediately from the equation

$$\cot \theta = i \coth(i\theta).$$

4. THE DERIVATIVE POLYNOMIALS FOR SECH AND SEC.

Define the polynomials $S_m(z)$ of degree $m = 0, 1, \dots$, by

$$\left(\frac{d}{d\theta}\right)^m \operatorname{sech} \theta = \operatorname{sech} \theta S_m(\tanh \theta). \quad (4.1)$$

One easily computes $S_0(z) = 1, S_1(z) = -z, S_2(z) = 2z - 1$, etc.

Proposition 3: For $m = 0, 1, \dots$,

$$S_m(z) = p_m\left(\frac{1+z}{-2}; 1, 2\right) \quad (4.2)$$

$$= \sum_{k=0}^m \binom{m}{k} 2^k \omega_k\left(\frac{z+1}{-2}\right), \text{ or}$$

$$S_m(z) = \sum_{j=0}^m \left[(-1)^j j! \sum_{k=j}^m \binom{m}{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} 2^{k-j} \right] (z+1)^j.$$

Proof: Starting from the representation

$$\operatorname{sech} \theta = \frac{2e^\theta}{1 + e^{2\theta}},$$

we apply Lemma 1 with $x = e^\theta$ to get

$$\begin{aligned} \left(\frac{d}{d\theta}\right)^m \operatorname{sech} \theta &= 2(xD)^m \frac{x}{1+x^2} = \frac{2x}{1+x^2} p_m \left(\frac{-x^2}{1+x^2}; 1, 2\right) \\ &= \operatorname{sech} \theta p_m \left(\frac{-e^{2\theta}}{1+e^{2\theta}}; 1, 2\right), \end{aligned}$$

which is (4.2), because

$$\frac{-e^{2\theta}}{1+e^{2\theta}} = \frac{-1}{2}(1 + \tanh \theta).$$

Remark: The numbers $E_m = S_m(0)$ are the *Euler numbers*, defined by

$$\operatorname{sech}(z) = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}$$

(see [1], [11, p. 153]). From (4.2) and (3.6):

$$\begin{aligned} E_m &= \sum_{k=0}^m \binom{m}{k} 2^k \omega_k \left(\frac{-1}{2}\right) = \sum_{k=0}^m \binom{m}{k} (-1)^k C_k(0), \text{ or} \\ E_m &= \sum_{k=0}^m \binom{m}{k} \frac{2^{k+1}}{k+1} (1 - 2^{k+1}) B_{k+1}, \text{ or} \\ E_m &= \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} 2^k (1 - 2^k) B_k \end{aligned} \tag{4.3}$$

cf. [11, p. 152].

Remark: $S_m(z)$ is also the derivative polynomial for the hyperbolic cosecant, i.e.,

$$\left(\frac{d}{d\theta}\right)^m \operatorname{csch} \theta = \operatorname{csch} \theta S_m(\coth \theta). \tag{4.4}$$

This can be easily seen by the same method using the representation

$$\operatorname{csch} \theta = \frac{-2e^\theta}{1 - e^{2\theta}},$$

or by looking at the formation of the derivatives. It is interesting that if one uses the alternative representation

$$\operatorname{csch} \theta = \frac{2e^{-\theta}}{1 - e^{-2\theta}} \quad (4.5)$$

for the above proof, the result is an equivalent form of $S_m(z)$.

We turn now to the derivative polynomials Q_m for the secant,

$$\left(\frac{d}{d\theta}\right)^m \sec \theta = \sec \theta Q_m(\tan \theta).$$

Proposition 4: For $m = 1, 2, \dots$ one has:

$$Q_m(z) = i^m S_m(iz). \quad (4.6)$$

The proof follows from (4.1) and the equation

$$\sec \theta = \operatorname{sech}(i\theta).$$

It is good to mention here also a convenient differentiation rule for the cosecant which follows from the representation

$$\frac{1}{\sin \theta} = \frac{1}{2} \left(\tan \frac{\theta}{2} + \cot \frac{\theta}{2} \right), \quad (4.7)$$

$$\left(\frac{d}{d\theta}\right)^m \frac{1}{\sin \theta} = \frac{1}{2^{m+1}} \left[P_m \left(\tan \frac{\theta}{2} \right) + (-1)^m P_m \left(\cot \frac{\theta}{2} \right) \right]. \quad (4.8)$$

5. THE HILBERT-EISENSTEIN SERIES AND THE REFLECTION FORMULA FOR THE POLYGAMMA FUNCTION

Consider the series

$$e_r(z) = \frac{1}{z^r} + \sum_{k=1}^{+\infty} \left(\frac{1}{(z+k)^r} + \frac{1}{(z-k)^r} \right),$$

when $r > 0$, or

$$e_r(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z+k)^r}$$

when $r > 1$. Its theory is nicely presented in Chapter 11 of Remmerts book [10], where also a biographical note on Ferdinand Gotthold Max Eisenstein (1823-1852) can be found. We shall evaluate this series using the tangent derivative polynomials. It is known that [10, p. 326]:

$$e_1(z) = \pi \cot \pi z. \tag{5.1}$$

Differentiating this equation $r - 1$ times we find according to (3.15)

$$(-1)^{r-1}(r-1)! e_r(z) = \pi \left(\frac{d}{dz}\right)^{r-1} \cot \pi z = -\pi^r P_{r-1}(-\cot \pi z), \tag{5.2}$$

and therefore, from (3.11),

$$e_r(z) = \frac{-i^r \pi^r}{(r-1)!} \csc^2 \pi z \sum_{k=1}^{r-1} \left\{ \begin{matrix} r-1 \\ k \end{matrix} \right\} 2^{r-k-1} k! (i \cot \pi z - 1)^{k-1}.$$

We can also write e_m explicitly as a polynomial of e_1 . From (5.1) and (5.2),

$$e_r = \frac{(-1)^r \pi^r}{(r-1)!} P_{r-1} \left(\frac{-1}{\pi} e_1 \right). \tag{5.3}$$

Next we look at the Digamma function

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) \tag{5.4}$$

where $\Gamma(z)$ is Euler's Gamma function. The Polygamma functions ψ_n are defined as the higher derivatives of ψ , see [1, 6.4], [12]. Thus, $\psi_0(z) = \psi(z)$ and

$$\psi_n(z) := \psi^{(n)}(z) = \left(\frac{d}{dz}\right)^{n+1} \log \Gamma(z)$$

($n = 1, 2, \dots$). The Digamma function satisfies the equation [1], [12, p. 14]:

$$\psi(z) - \psi(1 - z) = -\pi \cot \pi z,$$

(which reveals, in particular, that $\psi(z) - \psi(1 - z) = -e_1(z)$). Differentiating this equation n times, we find the explicit form of the *reflection formula* for the Polygamma function [1, p. 260], [12, p. 22], [14]:

$$\begin{aligned} \psi_n(z) - (-1)^n \psi_n(1 - z) &= \pi^{n+1} P_n(-\cot \pi z) \\ &= -i^{n+1} \pi^{n+1} \csc^2 \pi z \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} 2^{n-k} k! (i \cot \pi z - 1)^{k-1}. \end{aligned} \quad (5.5)$$

6. EVALUATION OF INTEGRALS

Hoffman presented in [5, Theorems 4.1 and 4.2] the following evaluations

$$\int_{-\infty}^{+\infty} \frac{x^n e^{ax}}{e^x + 1} dx = \pi^{n+1} \csc a\pi Q_n(-\cot a\pi), \quad n \geq 0,$$

$$\int_{-\infty}^{+\infty} \frac{x^n e^{ax}}{e^x - 1} dx = \pi^{n+1} P_n(-\cot a\pi), \quad n \geq 1,$$

where $0 < a < 1$. We want to add some more examples.

Example 1: Consider the integral [9, 2.5.46.5, p. 468]

$$\int_0^{\infty} \frac{\cos ax}{\cosh x} dx = \frac{\pi}{2} \operatorname{sech} \frac{\pi a}{2}, \quad (6.1)$$

for all real a . Differentiating this equation n times with respect to a we find

$$\int_0^{\infty} \frac{x^n}{\cosh x} \cos \left(ax + \frac{\pi n}{2} \right) dx = \left(\frac{\pi}{2} \right)^{n+1} \operatorname{sech} \frac{\pi a}{2} S_n \left(\tanh \frac{\pi a}{2} \right). \quad (6.2)$$

Example 2: Next we consider a similar integral [9, 2.5.46.4, p. 467]:

$$\int_0^{\infty} \frac{\sin ax}{\cosh x} dx = -\frac{\pi}{2} \tanh \frac{\pi a}{2} + \frac{i}{2} \left[\psi \left(\frac{1 - ia}{4} \right) - \psi \left(\frac{1 + ia}{4} \right) \right], \quad (6.3)$$

where $\psi(z)$ is the Digamma function (5.4). This leads to

$$\int_0^\infty \frac{x^n}{\cosh x} \sin\left(ax + \frac{\pi n}{2}\right) dx = -\left(\frac{\pi}{2}\right)^{n+1} C_n\left(\tanh \frac{\pi a}{2}\right) \quad (6.4)$$

$$+ \frac{i}{2^{2n+1}} \left[(-i)^n \psi^{(n)}\left(\frac{1-ia}{4}\right) - i^n \psi^{(n)}\left(\frac{1+ia}{4}\right) \right].$$

In the same way one obtains explicit evaluations of the integrals (2) and (5) in section 2.5.47, p. 469 in [9].

The two integrals (6.2) and (6.4) can be united in one to give a more compact result.

Proposition 5: For every $n = 0, 1, \dots$ and every real a ,

$$\int_0^\infty \frac{x^n}{\cosh x} e^{iax} dx = \quad (6.5)$$

$$(-1)^n i^n \left(\frac{\pi}{2}\right)^{n+1} \left[\operatorname{sech} \frac{\pi a}{2} S_n\left(\tanh \frac{\pi a}{2}\right) - i C_n\left(\tanh \frac{\pi a}{2}\right) \right]$$

$$+ \frac{1}{2^{2n+1}} \left[\psi^{(n)}\left(\frac{1+ia}{4}\right) - (-1)^n \psi^{(n)}\left(\frac{1-ia}{4}\right) \right].$$

This formula obviously extends to complex $|a| < 1$ (cf. 2.5.47(4), p. 469 in [9]).

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