PRODUCTS OF NUMBERS WHICH OBEY A FIBONACCI-TYPE RECURRENCE

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ABSTRACT

Let

$$Q_r(n) = G_{n+1}G_{n+2}\cdots G_{n+r} \tag{0.1}$$
$$\hat{Q}_r(n) = J_{n+1}J_{n+2}\cdots J_{n+r} \tag{0.2}$$

where, for various non-zero constants a, b, and c, one defines

$$G_m = aG_{m-1} + bG_{m-2} \tag{0.3}$$
$$J_m = aJ_{m-1} + bJ_{m-2} + cJ_{m-3}. \tag{0.4}$$

Through repeated iterations, one can show that

$$Q_r(n) = \sum_{j=1}^{r} R_j^r(a,b)G_{n+1}^{r+1-j}G_n^{j-1} \tag{0.5}$$

$$\hat{Q}_r(n) = \sum_{p=1}^{r-1}\sum_{q=1}^{r-p} R_{p,q}^r(a,b,c)J_{n+2}^pJ_{n+1}^qJ_n^{r-p-q}, \tag{0.6}$$

where the $R_j^r(a,b)$ and $R_{p,q}^r(a,b,c)$ are polynomials that obey a recurrence relation. This recurrence relation is a sum whose terms are binomial coefficients times monomials a^lb^k for (0.5) or binomial coefficients times monomials $a^lb^kc^s$ for (0.6).

1. INTRODUCTION

The Fibonacci numbers are one of the most studied combinatorial sequences. Various papers have been written about their properties. However, when perusing the literature, we found that no one has investigated the product of r consecutive Fibonacci numbers, namely the quantity $\tilde{P}_r(n) = F_{n+1}F_{n+2}\cdots F_{n+r}$, where F_n is the n^{th} Fibonacci number defined by $F_n = F_{n-1} + F_{n-2}$, $F_1 = 1$, $F_0 = 0$. By repeatedly iterating $\tilde{P}_r(n)$, we discovered that $\tilde{P}_r(n)$

can be written as a linear combination of products of the form $F_{n+1}^u F_n^v$, where $u + v = r$. The coefficients of this linear combination obey a recurrence relation that *only* depends on the Fibonacci recurrence $F_n = F_{n-1} + F_{n-2}$, and *not* on the initial conditions F_0 and F_1. This coefficient recurrence is recorded in Theorem 1. For example, the Lucas numbers, defined by $L_n = L_{n-1} + L_{n-2}, L_1 = 3, L_0 = 1$, are a sequence which obeys the Fibonacci recurrence, and hence, the coefficient recurrence of Theorem 1. By modifying the recurrence that defines F_n, we are able to form various generalizations of Theorem 1. These generalizations are given in Theorem 2, Theorem 3, and Theorem 4.

2. FIBONACCI RECURRENCE $\mathbf{H}_m = H_{m-1} + H_{m-2}$

Consider the product $P_r(n) = H_{n+1}H_{n+2}\cdots H_{n+r}$ of r consecutive numbers which obey the Fibonacci-type recurrence $H_m = H_{m-1} + H_{m-2}$. By applying this recurrence repeatedly, we find

$$\begin{aligned}
P_1(n) &= H_{n+1},\\
P_2(n) &= H_{n+1}^2 + H_{n+1}H_n,\\
P_3(n) &= 2H_{n+1}^3 + 3H_{n+1}^2 H_n + H_{n+1}H_n^2,\\
P_4(n) &= 6H_{n+1}^4 + 13H_{n+1}^3 H_n + 9H_{n+1}^2 H_n^2 + 2H_{n+1}H_n^3.
\end{aligned}$$

These calculations suggest the following formula for $P_r(n)$.

Theorem 1: *Let* $P_r(n) = H_{n+1}H_{n+2}\cdots H_{n+r}$. *Then,*

$$P_r(n) = \sum_{j=1}^{r} R_j^r H_{n+1}^{r+1-j} H_n^{j-1}, \tag{2.1}$$

where,

$$R_j^{r+1} = F_{r+1}R_j^r + F_r R_{j-1}^r, \qquad R_1^1 = 1, \tag{2.2}$$

and $R_j^r = 0$ *for* $j \le 0$ *or* $j > r$. *Note that* F_r *is the* r^{th} *Fibonacci number.*

Proof of Theorem 1: Mathematical induction on r. Clearly (2.1) is true when $r = 1$. We now assume (2.1) is true for arbitrary r and compute $P_{r+1}(n)$. By definition

$$P_{r+1}(n) = H_{n+r+1}P_r(n). \tag{2.3}$$

Using the induction hypothesis (2.1), along with the following well-known fact [1, p. 88],

$$H_{n+r} = F_r H_{n+1} + F_{r-1}H_n \tag{2.4}$$

we can simplify the right side of (2.3) to obtain

$$P_{r+1}(n) = \sum_{j=1}^{r} R_j^r (F_{r+1} H_{n+1}^{r+2-j} H_n^{j-1} + F_r H_{n+1}^{r+1-j} H_n^j) \tag{2.5}$$

$$= \sum_{j=1}^{r} R_j^r F_{r+1} H_{n+1}^{r+2-j} H_n^{j-1} + \sum_{j=2}^{r+1} F_r H_{n+1}^{r+2-j} H_n^{j-1} R_{j-1}^r \tag{2.6}$$

$$= \sum_{j=1}^{r+1} (F_{r+1} R_j^r + F_r R_{j-1}^r) H_{n+1}^{r+2-j} H_n^{j-1}. \tag{2.7}$$

Note that the parenthetical quantity of (2.7) is exactly (2.2). This proves our claim. □

In other words, Theorem 1 states that $P_r(n)$ may be expanded as a linear combination of products of the form $H_{n+1}^u H_n^v$, where $u + v = r$. Table 7.1 shows the first eight rows of the R_j^r triangle.

An interesting feature of the array is that the left diagonal and right column are the products of consecutive Fibonacci numbers, i.e.

$$R_1^r = \prod_{i=1}^{r} F_i = R_{r+1}^{r+1}.$$

Another interesting property is that the sums of the rows are also Fibonacci products, i.e.

$$\sum_{j=1}^{r} R_j^r = \prod_{i=1}^{r+1} F_i. \tag{2.8}$$

We prove (2.8) by mathematical induction on r and noting that (2.2) implies

$$\sum_{j=1}^{r+1} R_j^{r+1} = \sum_{j=1}^{r+1} F_r R_{j-1}^r + \sum_{j=1}^{r+1} F_{r+1} R_j^r.$$

A similar argument shows

$$\sum_{j=1}^{r} (-1)^j R_j^r = 0, \qquad for \qquad r \geq 1. \tag{2.9}$$

Equations (2.8) and (2.9) afford us a numerical check when calculating the coefficients in each row of Table 7.1.

These properties suggest that we consider the general polynomial

$$P_r(x,y) = \sum_{j=1}^{r} R_j^r x^{r+1-j} y^{j-1}. \tag{2.10}$$

By mathematical induction on r, it is easy to show that, in fact,

$$P_r(x,y) = x(x+y)(2x+y)(3x+2y)(5x+3y)\cdots(F_r x + F_{r-1}y). \tag{2.11}$$

Equation (2.11) implies that we may think of $P_r(x,y)$ as a kind of Fibonacci generalization of the ordinary binomial theorem.

It follows from (2.10) and (2.11) that

$$\sum_{j=1}^{r} R_j^r x^{j-1} = (x+1)(x+2)(2x+3)(3x+5)\cdots(F_{r-1}x + F_r). \tag{2.12}$$

The zeroes of the polynomial in (2.12) are, of course, negative ratios of consecutive Fibonacci numbers.

3. GENERALIZED FIBONACCI RECURRENCE: $\mathbf{G}_m = aG_{m-1} + bG_{m-2}$

We now consider the product $Q_r(n) = G_{n+1}G_{n+2}\cdots G_{n+r}$ of r consecutive numbers which obey the Fibonacci-type recurrence $G_m = aG_{m-1} + bG_{m-2}$. Note that a and b are fixed, non-zero constants. By applying this recurrence repeatedly, we find

$$\begin{aligned}
Q_1(n) &= G_{n+1}\\
Q_2(n) &= aG_{n+1}^2 + bG_{n+1}G_n\\
Q_3(n) &= (a^3+ab)G_{n+1}^3 + (2a^2b+b^2)G_{n+1}^2G_n + ab^2G_{n+1}G_n^2\\
Q_4(n) &= (a^6+3a^4b+2a^2b^2)G_{n+1}^4 + (3a^5b+7a^3b^2+3ab^3)G_{n+1}^3G_n\\
&\quad + (3a^4b^2+5a^2b^3+b^4)G_{n+1}^2G_n^2 + (a^3b^3+ab^4)G_{n+1}G_n^3.
\end{aligned}$$

These calculations suggest the following generalization of Theorem 1.

Theorem 2: *Let $Q_r(n) = G_{n+1}G_{n+2}\cdots G_{n+r}$. Then,*

$$Q_r(n) = \sum_{j=1}^{r} R_j^r(a,b) G_{n+1}^{r+1-j} G_n^{j-1}, \tag{3.1}$$

where,

$$R_j^{r+1}(a,b) = R_j^r(a,b)\sum_{k=0}^{[\frac{r}{2}]}\binom{r-k}{k}a^{r-2k}b^k + R_{j-1}^r(a,b)\sum_{k=0}^{[\frac{r-1}{2}]}\binom{r-1-k}{k}a^{r-1-2k}b^{k+1}, \quad (3.2)$$

and $R_1^1(a,b) = 1$, $R_j^r(a,b) = 0$ *for* $j \le 0$ *or* $j > r$.

Before we prove Theorem 2, we should note Remark 1. This remark, which parallels (2.4), is proven by an induction over r which utilizes Pascal's binomial identity. In particular,

Remark 1: *Let* G_{n+r} *be as previously defined. Then,*

$$G_{n+r} = G_{n+1}\sum_{k=0}^{[\frac{r-1}{2}]}\binom{r-k-1}{k}a^{r-2k-1}b^k + G_n\sum_{k=0}^{[\frac{r-2}{2}]}\binom{r-k-2}{k}a^{r-2k-2}b^{k+1}. \quad (3.3)$$

Proof of Theorem 2: Mathematical induction on r. First note that (3.1) is true when $r = 1$. Now assume (3.1) is true for arbitrary r and compute $Q_{r+1}(n)$. By definition, and the induction hypothesis (3.1),

$$Q_{r+1}(n) = G_{n+r+1}Q_r(n) = G_{n+r+1}\sum_{j=1}^{r}R_j^r(a,b)G_{n+1}^{r+1-j}G_n^{j-1}. \quad (3.4)$$

Substitute (3.3) into the right hand sum of (3.4) to obtain

$$Q_{r+1}(n) = \sum_{j=1}^{r}\sum_{k=0}^{[\frac{r}{2}]}R_j^r(a,b)a^{r-2k}b^k\binom{r-k}{k}G_{n+1}^{r+2-j}G_n^{j-1}$$

$$+\sum_{j=1}^{r}\sum_{k=0}^{[\frac{r-1}{2}]}R_j^r(a,b)a^{r-2k-1}b^{k+1}\binom{r-k-1}{k}G_{n+1}^{r+1-j}G_n^{j}$$

$$= \sum_{j=1}^{r}\sum_{k=0}^{[\frac{r}{2}]}R_j^r(a,b)a^{r-2k}b^k\binom{r-k}{k}G_{n+1}^{r+2-j}G_n^{j-1}$$

$$+\sum_{j=2}^{r+1}\sum_{k=0}^{[\frac{r-1}{2}]}R_{j-1}^r(a,b)a^{r-2k-1}b^{k+1}\binom{r-k-1}{k}G_{n+1}^{r+2-j}G_n^{j-1}$$

$$= \sum_{j=1}^{r+1}AG_{n+1}^{r+2-j}G_n^{j-1},$$

where,

$$A = R_j^r(a,b)\sum_{k=0}^{[\frac{r}{2}]}\binom{r-k}{k}a^{r-2k}b^k + R_{j-1}^r(a,b)\sum_{k=0}^{[\frac{r-1}{2}]}\binom{r-1-k}{k}a^{r-1-2k}b^{k+1}. \tag{3.5}$$

Since (3.5) is exactly (3.2), we have proven the theorem. □

4. TRIPLE RECURRENCE: $\mathbf{K}_m = K_{m-1} + K_{m-2} + K_{m-3}$

Next, we will consider product $\hat{P}_r(n) = K_{n+1}K_{n+2}\cdots K_{n+r}$ of r consecutive numbers which obey the basic linear three term recurrence, $K_m = K_{m-1} + K_{m-2} + K_{m-3}$. By iterating this recurrence repeatedly, we find that

$$\begin{aligned}
\hat{P}_1(n) &= K_{n+1}\\
\hat{P}_2(n) &= K_{n+2}K_{n+1}\\
\hat{P}_3(n) &= K_{n+2}^2K_{n+1} + K_{n+2}K_{n+1}^2 + K_{n+2}K_{n+1}K_n\\
\hat{P}_4(n) &= 2K_{n+2}^3K_{n+1} + 4K_{n+2}^2K_{n+1}^2 + 3K_{n+2}^2K_{n+1}K_n + 2K_{n+2}K_{n+1}^3\\
&\quad + 3K_{n+2}K_{n+1}^2K_n + K_{n+2}K_{n+1}K_n^2.
\end{aligned}$$

These calculations suggest a formula for $\hat{P}_r(n)$ similiar to that for $P_r(n)$. We record this formula in Theorem 3. The proof of Theorem 3 is omitted since the proof follows the procedure we used in proving Theorem 1. The only difference in the proof of Theorem 3 is that (2.4) becomes

$$K_{n+r} = \hat{F}_{r-1}K_{n+2} + \tilde{F}_{r-2}K_{n+1} + \hat{F}_{r-2}K_n, \tag{4.1}$$

where,

1. $\hat{F}_r = \hat{F}_{r-1} + \hat{F}_{r-2} + \hat{F}_{r-3}$, $\quad \hat{F}_0 = 0, \hat{F}_1 = 1, \hat{F}_2 = 1$.
2. $\tilde{F}_r = \tilde{F}_{r-1} + \tilde{F}_{r-2} + \tilde{F}_{r-3}$, $\quad \tilde{F}_0 = 0, \tilde{F}_1 = 1, \tilde{F}_2 = 2$.

Theorem 3: *Let $\hat{P}_r(n) = K_{n+1}K_{n+2}\cdots K_{n+r}$. Then,*

$$\hat{P}_r(n) = \sum_{k=1}^{r-1}\sum_{l=1}^{r-k} R_{k,l}^r K_{n+2}^k K_{n+1}^l K_n^{r-k-l}, \qquad r \geq 2, \tag{4.2}$$

where,

$$R_{k,l}^{r+1} = \hat{F}_r R_{k-1,l}^r + \tilde{F}_{r-1}R_{k,l-1}^r + \hat{F}_{r-1}R_{k,l}^r. \tag{4.3}$$

For a fixed r, $R_{k,l}^r$ is identically zero when k and l are outside the range of summation provided by (4.2).

5. GENERALIZED TRIPLE RECURRENCE: $\mathbf{J}_m = aJ_{m-1} + bJ_{m-2} + cJ_{m-3}$

Consider the product $\hat{Q}_r(n) = J_{n+1}J_{n+2}\cdots J_{n+r}$ of r consecutive numbers which obey the generalized form of the three term linear recurrence, $J_m = aJ_{m-1} + bJ_{m-2} + bJ_{m-3}$. In this case, a, b, and c are fixed non-zero constants. Through repeated iterations, we find

$$\begin{aligned}
\hat{Q}_1(n) &= J_{n+1}\\
\hat{Q}_2(n) &= J_{n+2}J_{n+1}\\
\hat{Q}_3(n) &= aJ_{n+2}^2J_{n+1} + bJ_{n+2}J_{n+1}^2 + cJ_{n+2}J_{n+1}J_n\\
\hat{Q}_4(n) &= (a^3+ba)J_{n+2}^3J_{n+1} + (2a^2b+b^2+ca)J_{n+2}^2J_{n+1}^2 + (2a^2c+bc)J_{n+2}^2J_{n+1}J_n\\
&\quad + (ab^2+cb)J_{n+2}J_{n+1}^3 + (2abc+c^2)J_{n+2}J_{n+1}^2J_n + ac^2J_{n+2}J_{n+1}J_n^2.
\end{aligned}$$

These calculations suggest the following generalization of Theorem 3, which we record as Theorem 4. We omit the proof of Theorem 4 since it parallels that of Theorem 3. The only difference is that (3.3) becomes

$$J_{n+r} = C_r^2J_{n+2} + C_r^1J_{n+1} + C_r^0J_n, \tag{5.1}$$

where,

$$C_r^2 = \sum_{j=0}^{[\frac{r-2}{3}]}\sum_{k=0}^{[\frac{r-3j-2}{2}]}\binom{j+k}{j}\binom{r-2-2j-k}{j+k}a^{r-2-3j-2k}b^kc^j.$$

$$C_r^1 = \sum_{j=0}^{[\frac{r-1}{3}]}\sum_{k=0}^{[\frac{r-3j-1}{2}]}\binom{j+k}{j}\binom{r-2-2j-k}{j+k-1}a^{r-1-3j-2k}b^kc^j.$$

$$C_r^0 = cC_{r-1}^2.$$

Theorem 4: *Let $\hat{Q}_r(n) = J_{n+1}J_{n+2}\cdots J_{n+r}$. Then,*

$$\hat{Q}_r(n) = \sum_{p=1}^{r-1}\sum_{q=1}^{r-p}R_{p,q}^r(a,b,c)J_{n+2}^pJ_{n+1}^qJ_n^{r-p-q}, \qquad r \geq 2, \tag{5.2}$$

where,

$$R_{p,q}^{r+1}(a,b,c) = \sum_{j=0}^{[\frac{r-1}{3}]} \sum_{k=0}^{[\frac{r-3j-1}{2}]} R_{p-1,q}^{r}(a,b,c) \binom{j+k}{j} \binom{r-1-2j-k}{j+k} a^{r-1-3j-2k} b^k c^j$$

$$+ \sum_{j=0}^{[\frac{r}{3}]} \sum_{k=0}^{[\frac{r-3j}{2}]} R_{p,q-1}^{r}(a,b,c) \binom{j+k}{j} \binom{r-1-2j-k}{j+k-1} a^{r-3j-2k} b^k c^j$$

$$+ \sum_{j=0}^{[\frac{r-2}{3}]} \sum_{k=0}^{[\frac{r-3j-2}{2}]} R_{p,q}^{r}(a,b,c) \binom{j+k}{j} \binom{r-2-2j-k}{j+k} a^{r-2-3j-2k} b^k c^{j+1}$$

For a fixed r, $R_{p,q}^r(a,b,c)$ is identically zero when p and q are outside the range of summation provided by (5.2).

6. OPEN QUESTIONS

In this paper, we have analyzed the quantity $\tilde{P}_r(n) = F_n F_{n+1} \cdots F_{n+r}$, where F_n is defined by either a two term or three term Fibonacci-type recurrence. Such analysis led to the formation of polynomial coefficients $R_j^r(a,b)$ and $R_{p,q}^r(a,b,c)$ which obey the relationships defined in Theorem 2 and Theorem 4, respectively. Future research should investigate the various properties of these polynomials. Furthermore, an interesting research problem involves extending the iteration analysis used throughout this paper to the product of r numbers, each of which is defined by a k term Fibonacci-type recurrence, for $k \geq 4$.

7. TABLES

1							
1	1						
2	3	1					
6	13	9	2				
30	83	84	37	6			
240	814	1087	716	233	30		
3120	12502	20643	18004	8757	2254	240	
65520	303102	596029	646443	417949	161175	34342	3120

Table 7.1: The first eight rows of the R_j^r triangle.
The row index corresponds to r, where $1 \leq r \leq 8$,
while the column index corresponds to j, $1 \leq j \leq 8$.
For example, $R_2^4 = 13$ and $R_3^5 = 84$.

$R_1^1(a,b) = 1$

$R_1^2(a,b) = a \qquad R_2^2(a,b) = b$

$R_1^3(a,b) = a^3 + ab \qquad R_2^3(a,b) = 2a^2b + b^2 \qquad R_3^3(a,b) = ab^2$

$R_1^4(a,b) = a^6 + 3a^4b + 2a^2b^2 \qquad R_2^4(a,b) = 3a^5b + 7a^3b^2 + 3ab^3$

$R_3^4(a,b) = 3a^4b^2 + 5a^2b^3 + b^4$

$R_4^4(a,b) = a^3b^3 + ab^4$

$R_1^5(a,b) = a^{10} + 6a^8b + 12a^6b^2 + 9a^4b^3 + 2a^2b^4$

$R_2^5(a,b) = 4a^9b + 21a^7b^2 + 35a^5b^3 + 20a^3b^4 + 3ab^5$

$R_3^5(a,b) = 6a^8b^2 + 27a^6b^3 + 36a^4b^4 + 14a^2b^5 + b^6$

$R_4^5(a,b) = 4a^7b^3 + 15a^5b^4 + 15a^3b^5 + 3ab^6 \qquad R_5^5(a,b) = a^6b^4 + 3a^4b^5 + 2a^2b^6$

$R_1^6(a,b) = a^{15} + 10a^{13}b + 39a^{11}b^2 + 75a^9b^3 + 74a^7b^4 + 35a^5b^5 + 6a^3b^6$

$R_2^6(a,b) = 5a^{14}b + 46a^{12}b^2 + 162a^{10}b^3 + 274a^8b^4 + 229a^6b^5 + 87a^4b^6 + 11a^2b^7$

$R_3^6(a,b) = 10a^{13}b^2 + 84a^{11}b^3 + 264a^9b^4 + 385a^7b^5 + 263a^5b^6 + 75a^3b^7 + 6ab^8$

$R_4^6(a,b) = 10a^{12}b^3 + 76a^{10}b^4 + 210a^8b^5 + 257a^6b^6 + 136a^4b^7 + 26a^2b^8 + b^9$

$R_5^6(a,b) = 5a^{11}b^4 + 34a^9b^5 + 81a^7b^6 + 80a^5b^7 + 30a^3b^8 + 3ab^9$

$R_6^6(a,b) = a^{10}b^5 + 6a^8b^6 + 12a^6b^7 + 9a^4b^8 + 2a^2b^9$

Table 7.2: $R_j^r(a,b)$, where $1 \le r \le 6$.
Note that if $a = 1 = b$, we recapture the first six rows of Table 7.1.

$R_{0,1}^1 = 1$

$R_{1,1}^2 = 1$

$R_{2,1}^3 = 1 \qquad R_{1,2}^3 = 1 \qquad R_{1,1}^3 = 1$

$R_{3,1}^4 = 2 \qquad R_{2,2}^4 = 4 \qquad R_{2,1}^4 = 3 \qquad R_{1,3}^4 = 2 \qquad R_{1,2}^4 = 3 \qquad R_{1,1}^4 = 1$

$R_{4,1}^5 = 8 \qquad R_{3,2}^5 = 22 \qquad R_{3,1}^5 = 16 \qquad R_{2,3}^5 = 20 \qquad R_{2,2}^5 = 29 \qquad R_{2,1}^5 = 10$

$R_{1,4}^5 = 6 \qquad R_{1,3}^5 = 13 \qquad R_{1,2}^5 = 9 \qquad R_{1,1}^5 = 2$

$R_{5,1}^6 = 56 \qquad R_{4,2}^6 = 202 \qquad R_{4,1}^6 = 144 \qquad R_{3,3}^6 = 272 \qquad R_{3,2}^6 = 387 \qquad R_{3,1}^6 = 134$

$R_{2,4}^6 = 162 \qquad R_{2,3}^6 = 345 \qquad R_{2,2}^6 = 239 \qquad R_{2,1}^6 = 54$

$R_{1,5}^6 = 36 \qquad R_{1,4}^6 = 102 \qquad R_{1,3}^6 = 106 \qquad R_{1,2}^6 = 48 \qquad R_{1,1}^6 = 8$

Table 7.3: Non-zero values for $R_{k,l}^j$ when $1 \le j \le 6$.

$$R^1_{0,1}(a,b,c) = 1$$
$$R^2_{1,1}(a,b,c) = 1$$
$$R^3_{2,1}(a,b,c) = a \qquad R^3_{1,2}(a,b,c) = b \qquad R^3_{1,1}(a,b,c) = c$$
$$R^4_{3,1}(a,b,c) = a^3 + ab \qquad R^4_{2,2}(a,b,c) = 2a^2b + b^2 + ac \qquad R^4_{2,1}(a,b,c) = 2a^2c + bc$$
$$R^4_{1,3}(a,b,c) = ab^2 + bc \qquad R^4_{1,2}(a,b,c) = 2abc + c^2 \qquad R^4_{1,1}(a,b,c) = ac^2$$
$$R^5_{4,1}(a,b,c) = a^3c + abc + 2a^2b^2 + a^6 + 3a^4b$$
$$R^5_{3,2}(a,b,c) = ac^2 + b^2c + 5a^2bc + 7a^3b^2 + 3a^5b + 2a^4c + 3ab^3$$
$$R^5_{3,1}(a,b,c) = 2a^2c^2 + 3ab^2c + bc^2 + 7a^3bc + 3a^5c$$
$$R^5_{2,3}(a,b,c) = 4a^3bc + bc^2 + a^2c^2 + 5a^2b^3 + b^4 + 5ab^2c + 3a^4b^2$$
$$R^5_{2,2}(a,b,c) = 4a^3c^2 + c^3 + 6a^4bc + 6abc^2 + 2b^3c + 10a^2b^2$$
$$R^5_{2,1}(a,b,c) = 5a^2bc^2 + b^2c^2 + ac^3 + 3a^4c^2$$
$$R^5_{1,4}(a,b,c) = ab^4 + a^3b^3 + abc^2 + 2a^2b^2c + b^3c$$
$$R^5_{1,3}(a,b,c) = 3ab^3c + 4a^2bc^2 + 2b^2c^2 + ac^3 + 3a^3b^2c$$
$$R^5_{1,2}(a,b,c) = 2a^2c^3 + 3a^3bc^2 + bc^3 + 3ab^2c^2 \qquad R^5_{1,1}(a,b,c) = abc^3 + a^3c^3$$

Table 7.4: $R^j_{k,l}(a,b,c)$, where $1 \le j \le j$.
Note that if $a = 1 = b = c$, we recapture Table 7.3.

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