# THE EQUATION $m^2 - 4k = 5n^2$ AND UNIQUE REPRESENTATIONS OF POSITIVE INTEGERS

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## ABSTRACT

If $n$ is a positive integer, there exists one and only one pair $(j,k)$ of positive integers such that $(j+k+1)^2 - 4k = 5n^2$. The resulting unique representation of $n$ can be used to generate both the Wythoff difference array and the Fraenkel array. It also provides the solution of the complementary equation $b(n) = a(jn) + kn$ in all cases in which $a$ and $b$ are a pair of Beatty sequences and $a(n)$ is of the form $[rn]$ for $r$ an irrational number in the field $Q(\sqrt{5})$.

## 1. INTRODUCTION

Given a positive integer $n$, the Diophantine equation $m^2 - 4k = 5n^2$ has many solutions $(m,k)$. However, putting $m = j+k+1$ yields an equation that has one and only one solution $(j,k)$ for which both $j$ and $k$ are positive integers. After proving this in Section 2, we shall show, in Sections 3 and 4, how each of two recently introduced arrays can be generated by taking the pairs $(j,k)$ in a certain order. The arrays, called the Wythoff difference array (WDA) and the Fraenkel array, are defined just below. In Section 5, Gessel's Theorem regarding all the solutions of the equations $m^2 \pm 4 = 5n^2$ is generalized in connection with the WDA and the Fraenkel array. In Section 6, Theorem 1 is applied to the complementary equation $b(n) = a(jn) + kn$.

Throughout, the symbols $j,k$ are integers, and $n,g,h$ are positive integers. The golden number $(1+\sqrt{5})/2$ is denoted by $\tau$. The Fibonacci numbers $F_g$ and Lucas numbers $L_g$ are defined as usual by

$$F_0 = 0, \quad F_1 = 1, \quad F_g = F_{g-1} + F_{g-2} \quad \text{for } g \geq 2,$$
$$L_0 = 2, \quad L_1 = 1, \quad L_g = L_{g-1} + L_{g-2} \quad \text{for } g \geq 2.$$

The WDA, $\mathcal{D} = \{d(g,h)\}$, is given [3] by

$$d(g,h) = \lfloor g\tau \rfloor F_{2h-1} + (g-1)F_{2h-2}$$

and satisfies the following recurrence for rows:

$$d(g,h) = 3d(g,h-1) - d(g,h-2) \quad \text{for } h \geq 3.$$

The lower Wythoff sequence (indexed as A000201 in [5]) and upper Wythoff sequence (A001950) are given by $\{\lfloor n\tau \rfloor\}$ and $\{\lfloor n\tau^2 \rfloor\}$, respectively. As the dispersion [6] of the upper Wythoff sequence, the WDA contains every $n$ exactly once. The northwest corner of $\mathcal{D}$ is shown here:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 13 | 34 | 89 | 233 | ... |
| 3 | 7 | 18 | 47 | 123 | 322 | 843 | |
| 4 | 10 | 26 | 68 | 178 | 466 | 1220 | |
| 6 | 15 | 39 | 102 | 267 | 299 | 1830 | |
| 8 | 20 | 52 | 136 | 356 | 932 | 2440 | |
| 9 | 23 | 60 | 157 | 411 | 1076 | 2817 | |
| 11 | 28 | 73 | 191 | 500 | 1309 | 3427 | |
| 12 | 31 | 81 | 212 | 555 | 1453 | 3804 | |
| 14 | 36 | 94 | 246 | 644 | 1686 | 4414 | |
| ⋮ | | | | | | | |

**Table 1. The Wythoff difference array**

The Fraenkel array, $\mathcal{F} = \{f(g,h)\}$, is introduced in [1] in connection with a combinatorial game and a numeration system. The number in row $g$ and column $h$ is

$$f(g,h) = \lfloor (g-1)\tau + 1 \rfloor F_{2h-1} + gF_{2h-2}.$$

This array satisfies the same row recurrence that $\mathcal{D}$ does:

$$f(g,h) = 3f(g,h-1) - f(g,h-2) \quad \text{for } h \geq 3.$$

(In [1], the array has an initial row consisting entirely of zeros.) The array $\mathcal{F}$ is the dispersion of the sequence $\{\lfloor n\tau^2 \rfloor + 1\}$, alias $\{\lfloor n\tau \rfloor + n + 1\}$, formed by adding 1 to the terms of the upper Wythoff sequence.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 8 | 21 | 55 | 144 | 377 | ... |
| 2 | 6 | 16 | 42 | 110 | 288 | 754 | |
| 4 | 11 | 29 | 76 | 199 | 521 | 1364 | |
| 5 | 14 | 37 | 97 | 254 | 665 | 1741 | |
| 7 | 19 | 50 | 131 | 343 | 898 | 2351 | |
| 9 | 24 | 63 | 165 | 432 | 1131 | 2961 | |
| 10 | 27 | 71 | 186 | 487 | 1275 | 3338 | |
| 12 | 32 | 84 | 220 | 576 | 1508 | 3948 | |
| 13 | 35 | 92 | 241 | 632 | 1652 | 4225 | |
| ⋮ | | | | | | | |

**Table 2. The Fraenkel array**

## 2. MAIN THEOREM

**Theorem 1**: *For every $n \geq 1$, there exists exactly one pair $j \geq 1$, $k \geq 1$ such that*

$$n = \sqrt{\frac{(j+k+1)^2 - 4k}{5}}\,. \tag{1}$$

*Explicitly,*

$$j = n^2 + n\lfloor n\tau \rfloor - \lfloor n\tau \rfloor^2, \tag{2}$$

$$k = \lfloor n\tau \rfloor^2 + (2-n)\lfloor n\tau \rfloor - n^2 - n + 1. \tag{3}$$

In order to prove Theorem 1, we first introduce notation and lemmas. For $n \geq 1$, let

$$k(n) = \lfloor n\tau \rfloor^2 + (2-n)\lfloor n\tau \rfloor - n^2 - n + 1 \tag{4}$$

$$m(n) = 2\lfloor n\tau \rfloor - n + 2 \tag{5}$$

$$i(n) = \begin{cases} (n-1)/2 - \lfloor n\tau \rfloor & \text{if } n \text{ is odd,} \\ n/2 - \lfloor n\tau \rfloor & \text{if } n \text{ is even.} \end{cases}$$

In the lemmas, $n$ stays fixed, and we abbreviate

$$\lfloor n\tau \rfloor \text{ as } x, \qquad m(n) \text{ as } m, \qquad k(n) \text{ as } k_1.$$

Define

$$m_i = m + 2i - 2 \quad \text{for } i \geq 1, \tag{6}$$

$$k_i = k_1 + (i-1)m + (i-1)^2. \tag{7}$$

**Lemma 1**: $4k_1 + 5n^2$ is a square:

$$4k_1 + 5n^2 = m^2.$$

**Proof**:

$$\begin{aligned} 4k_1 + 5n^2 &= 4[x^2 + (2-n)x - n^2 - n + 1] + 5n^2 \\ &= 4x^2 + n^2 + 4 - 4nx + 8x - 4n \\ &= (2x - n + 2)^2. \end{aligned}$$

**Lemma 2**: The least integer $i$ for which $4k_i + 5n^2$ is a nonnegative square is $i(n)$.

**Proof**: Let

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

We solve the equation $4k_i + 5n^2 = \delta_n$ for $i$:

$$\begin{aligned} \delta_n &= 4[k_1 + (i-1)m + (i-1)^2] + 5n^2 \\ &= 4k_1 + 5n^2 + 4(i-1)m + 4(i-1)^2 \\ &= m^2 + 4(i-1)m + 4(i-1)^2 \\ &= [m + 2(i-1)]^2, \end{aligned}$$

so that

$$\delta_n = \pm(2x - n + 2i).$$

If $n$ is even, write $n = 2q$ and find that $2\lfloor n\tau \rfloor - 2q + 2i = 0$, which yields $i = n/2 - \lfloor n\tau \rfloor$. If $n$ is odd, write $n = 2q - 1$, and find, using $\delta_n = -1$, that $i = (n-1)/2 - \lfloor n\tau \rfloor$.

**Lemma 3**: If $i \geq i(n)$, then

$$4k_i + 5n^2 = m_i^2.$$

**Proof**: Using Lemma 1, we have

$$\begin{aligned} 4k_i + 5n^2 &= 4[k_1 + (i-1)m + (i-1)^2] + 5n^2 \\ &= 4k_1 + 5n^2 + 4(i-1)m + 4(i-1)^2 \\ &= m_i^2. \end{aligned}$$

**Lemma 4**: If $k$ is an integer for which $4k + 5n^2$ is a square, then $k$ is one of the numbers $k_i$, where $i \geq i(n)$.

**Proof**: Suppose $4k + 5n^2$ is a square, $M^2$. Then $M$ has the same parity as $n$, so that $M$ must be one of the list, exhaustive by Lemma 2, of consecutive same-parity squares given by Lemma 1.

**Lemma 5**: $k_1 \geq 1$.

**Proof**: Let $f$ be the fractional part of $n\tau$, so that

$$0 < f = n\tau - \lfloor n\tau \rfloor < 1.$$

Then

$$\begin{aligned} k_1 &= \lfloor n\tau \rfloor^2 + (2-n)\lfloor n\tau \rfloor - n^2 - n + 1 \\ &= (1-f)[(1-f) + (2\tau - 1)n] \\ &\geq 1. \end{aligned}$$

**Lemma 6**: $k_0 \leq -1$.

**Proof**:

$$\begin{aligned} k_0 &= k_1 - m + 1 \\ &= k_1 - (n\lfloor n\tau \rfloor - n + 1) + 1 \\ &= (-\sqrt{5}n + f)f \\ &\leq -1. \end{aligned}$$

**Lemma 7**: If $i \geq 2$, then $m_i \leq k_i$.

**Proof**:

$$\begin{aligned} m_i &= m + 2i - 2 \\ &\le (i-1)m + (i-1)^2 \\ &\le k_i. \end{aligned}$$

by Lemma 5 and (6).

We now prove Theorem 1. Let $k = k_1$ and $j = m - k_1 - 1$, so that $m = j + k + 1$. By Lemma 1, $4k + 5n^2 = m^2$. By Lemmas 4 and 6, if $\widehat{k} < k$ and $4\widehat{k} + 5n^2$ is a square, then $k \le -1$. By Lemmas 4 and 5, if $\widehat{k} > k$ and $4\widehat{k} + 5n^2$ is a square, then $j \le -1$. Therefore $k$ and $j$ are the only pair of positive integers $\widehat{k}$ and $\widehat{j}$ that satisfy

$$(\widehat{j} + \widehat{k} + 1)^2 - 4\widehat{k} = 5n^2,$$

hence, they provide the unique solution of (1).

## 3. THE WYTHOFF DIFFERENCE ARRAY

Regarding equation (1), we now ask, for any fixed $j$, this question: *what values of $n$ are generated?* The answer is surprising and simple: *a row of the WDA*. We shall prove that as $j$ ranges through a certain nonincreasing sequence, the corresponding rows are the consecutive rows of the WDA. To that end, suppose $g \ge 1$ and $h \ge 1$, and abbreviate $\lfloor g\tau \rfloor$ as $x$. Let

$$j = j(g) = x^2 - (g-1)x - (g-1)^2,$$

$$k = k(g,h) = g^2 + (g - 1 + L_{2h-1})x - x^2 + (g-1)(L_{2h-2} - 2), \tag{8}$$

$$n = n(g,h) = xF_{2h-1} + (g-1)F_{2h-2}. \tag{9}$$

In view of (2), what we wish to prove is that

$$(j + k + 1)^2 - 4k = 5n^2. \tag{10}$$

We have

$$j + k + 1 = L_{2h-1} + (g-1)L_{2h-2} + 2,$$

so that, abbreviating $L_{2h-1}$ as $\widehat{L}_1$ and $L_{2h-2}$ as $\widehat{L}_2$,

$$\begin{aligned} (j+k+1)^2 - 4k &= (x\widehat{L}_1 + (g-1)\widehat{L}_2 + 2)^2 \\ &\quad - 4(g^2 + (g - 1 + \widehat{L}_1)x - x^2 + (g-1)(\widehat{L}_2 - 2)) \\ &= 8g + 4x - 4gx - 2x\widehat{L}_1\widehat{L}_2 + 2gx\widehat{L}_1\widehat{L}_2 \\ &\quad - 4g^2 + 4x^2 + \widehat{L}_2^2 - 2g\widehat{L}_2^2 + g^2\widehat{L}_2^2 + x^2\widehat{L}_1^2 - 4 \\ &= x^2\widehat{L}_1^2 + 2x\widehat{L}_1\widehat{L}_2(g-1) + \widehat{L}_2^2(g-1)^2 \\ &\quad + 8g + 4x - 4gx - 4g^2 + 4x^2 - 4. \end{aligned}$$

Now using the well-known identities

$$\widehat{L}_1^2 = L_{4h-4} + L_{4h-3} - 2, \tag{11}$$

$$\widehat{L}_2^2 = L_{4h-4} + 2, \tag{12}$$

$$\widehat{L}_1\widehat{L}_2 = L_{4h-3} + 1 \tag{13}$$

in (9), we have

$$\begin{aligned}(j+k+1)^2 - 4k &= x^2(L_{4h-4} + L_{4h-3} - 2) + 2x(g-1)(L_{4h-3}+1)\\ &\quad + (g-1)^2(L_{4h-4}+2)\\ &\quad + 8g + 4x - 4gx - 4g^2 + 4x^2 - 4\\ &= L_{4h-3}[x^2 + 2x(g-1)] + L_{4h-4}[x^2 + (g-1)^2] \qquad (14)\\ &\quad + 2[x^2 - (g-1)x - (g-1)^2]. \qquad (15)\end{aligned}$$

Thus, the left-hand side of (10) is expressed in terms of the Lucas numbers $L_{4h-4}$ and $L_{4h-3}$. We shall next convert the right-hand side of (8) to the same expression:

$$\begin{aligned}5n^2 &= 5[xF_{2h-1} + (g-1)F_{2h-2}]^2\\ &= 5x^2F_{2h-1}^2 + 10x(g-1)F_{2h-1}F_{2h-2} + 5(g-1)^2F_{2h-2}^2.\end{aligned}$$

Applying the well-known identities

$$5F_{2h-1}^2 = L_{4h-4} + L_{4h-3} + 2, \tag{16}$$
$$5F_{2h-2}^2 = L_{4h-4} - 2, \tag{17}$$
$$5F_{2h-1}F_{2h-2} = L_{4h-3} - 1, \tag{18}$$

gives

$$5n^2 = x^2(L_{4h-4} + L_{4h-3} + 2) + 2x(g-1)(L_{4h-3} - 1) + (g-1)^2(L_{4h-4} - 2),$$

as in (14) and (15), so that (10) is now proved

## 4. THE FRAENKEL ARRAY

In Section 3, we showed a method of generating by rows; to summarize, for each $j = j(g)$ for which (1), and hence (8), has a solution, the corresponding numbers $k = k(g,h)$ and $n = n(g,h)$ were generated. In this section, we shall reverse the roles of $j$ and $k$. That is, for each $k = k(g)$, we recognize all the $j(g,h)$ and thus generate a row of numbers $n(g,h)$. For the sake of analogy, we use the same notation as in Section 2; however, as functions, $j, k$, and $n$ are not the same as in Section 2. Moreover, in this section, the symbol $x$ abbreviates $\lfloor (g-1)\tau \rfloor$ rather than $\lfloor g\tau \rfloor$.

Let

$$k = k(g) = (x + g + 1)g - (x - 1)^2,$$
$$j = j(g, h) = x^2 + (L_{2h-1} + 2 - g)x - g^2 + (L_{2h-2} - 1)g + L_{2h-1}, \quad (19)$$
$$n = n(g, h) = (x + 1)F_{2h-1} + gF_{2h-2}. \quad (20)$$

Then

$$j + k + 1 = (x + 1)L_{2h-1} + gL_{2h-2},$$

and

$$(j + k + 1)^2 - 4k = (x + 1)^2 L_{2h-1}^2 + 2g(x + 1)L_{2h-2}L_{2h-1} + g^2 L_{2h-2}^2 - 4[(x + g + 1)g - (x + 1)^2].$$

Applying (11)-(13) and simplifying give

$$(j + k + 1)^2 - 4k = L_{4h-3}(x + g + 1)L_{4h-4}(x + 1)^2 + 2[(x + 1)^2 + g(1 - x - g)].$$

Meanwhile,

$$5n^2 = 5[(x + 1)F_{2h-1} + gF_{2h-2}]^2,$$

which, using (16)-(18), simplifies to the expression already obtained for $(j + k + 1)^2 - 4k$.

## 5. GESSEL'S THEOREM GENERALIZED

Gessel's Theorem [2] can be stated in two parts:

The solutions of $5n^2 + 4 = m^2$ are the pairs $(m, n) = (L_{2h}, F_{2h})$;
the solutions of $5n^2 - 4 = m^2$ are the pairs $(m, n) = (L_{2h-1}, F_{2h-1})$.

Recall from Section 4 that row 1 of the Fraenkel array is generated from (10) with $k = 1$, as $j$ runs through the numbers $L_{2h} - 2$, so that $m$ runs through the alternating Lucas numbers $L_{2h}$. The fact that row 1 consists of the numbers $F_{2h}$ in increasing order implies the first part of Gessel's theorem. Row 2 of the Fraenkel array corresponds to $g = 2$ and $k = 4$, for which $j$ takes the values $1, 9, 31, \ldots$, $m$ takes the values $6, 14, 36, \ldots$, $n$ takes the values $2, 6, 16, \ldots$, and the pairs $(m, n) = (2L_h, 2F_h)$ solve the equation $5n^2 + 16 = m^2$. In like manner, row 3 gives the solutions of the equation $5n^2 + 20 = m^2$. In general, row $g$ of the Fraenkel array gives the solutions of the equation

$$5n^2 + 4k(g) = m^2,$$

where $k(g)$ is as given by (4).

Recall from Section 3 that row 1 of the WDA is generated from (10) with $j = 1$, as $k$ runs through the alternating Lucas numbers $L_{2h-1}$. The fact that row 1 consists of the numbers $F_{2h-1}$ in increasing order constitutes a proof of the first part of Gessel's theorem, quite different from the proofs in [2]. We shall show here that the other rows of the WDA correspond to complete solutions of equations similar to $5n^2 - 4 = m^2$. It is easy to check that equation (10) is equivalent to

$$\begin{aligned} 5n^2 - 4j &= (j + k - 1)^2 \\ &= (m - 2)^2, \end{aligned}$$

so that row $g$ of the WDA gives the solutions of the equation

$$5n^2 - 4j(g) = (m - 2)^2,$$

where $j(g)$ is as given by (2) (with $g$ substituted for $n$). For example, row 4, giving solutions of the equation $5n^2 + 36 = (m - 2)^2$ consists of the numbers $n = n(h)$:

$$6, 15, 39, 102, \ldots ;$$

using $j = 9$, we find the numbers $m(h)$:

$$14, 35, 89, 230, \ldots ,$$

given by $m(h) = 2 + 3L_{2h+1}$.

## 6. THE COMPLEMENTARY EQUATION $b(n) = a(jn) + kn$

As in [4], under the assumption that sequences $a$ and $b$ partition the sequence of positive integers, the designation *complementary equations* applies to equations such as $b(n) = a(jn) + kn$, where $j$ and $k$ are fixed positive integers. For example, the solutions of the equation $b(n) = a(n) + n$ is the sequence $b$ given by $b(n) = \lfloor n\tau^2 \rfloor$, or equivalently, by $a(n) = \lfloor n\tau \rfloor$. It is shown in [4] that the equation $b(n) = a(jn) + kn$ is solved by a pair of Beatty sequences

$$a(n) = \lfloor rn \rfloor, \quad b(n) = \lfloor sn \rfloor,$$

where $r$ and $s$ are determined as follows: let

$$p = \frac{j - k + 1}{2}, \tag{21}$$

$$\sqrt{q} = \frac{\sqrt{(j + k + 1)^2 - 4k}}{2}.$$

Then

$$r = \frac{p + \sqrt{q}}{j} \quad \text{and} \quad s = \frac{j\sqrt{q} + q + jp - p^2}{q - (p - j)^2},$$

where $r$ and $s$ are related by

$$\frac{1}{r} + \frac{1}{s} = 1.$$

In this section, we seek those pairs $(j,k)$ for which $r$ has the form $c+d\sqrt{5}$, where $c$ and $d$ are rational and $d \neq 0$. In view of (21), the problem is essentially solved in Section 2, with solutions given by (2) and (3). For example, for $(j,k)=(1,4)$, we have

$$r = -1+\sqrt{5} \quad \text{and} \quad s = 3+\sqrt{5}.$$

For $(j,k)=(5,1)$, we have

$$r = \frac{5+3\sqrt{5}}{10} \quad \text{and} \quad s = \frac{7+3\sqrt{5}}{2},$$

and for $(j,k)=(4,5)$,

$$r = \frac{\sqrt{5}}{2} \quad \text{and} \quad s = 5+2\sqrt{5}.$$

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