RESTRICTED OCCUPANCY OF S KINDS OF CELLS AND GENERALIZED PASCAL TRIANGLES

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(Submitted April 2007 - Final Revision November 2007)

ABSTRACT

There are several well-known formulas counting the number of distinct allocations of \( n \) indistinguishable objects into \( m \) distinguishable cells, each of which has capacity \( k - 1 \). In the present paper we generalize four of them by relaxing the assumption that each of the \( m \) cells has capacity \( k - 1 \) and assuming instead that there are \( s \) kinds of cells and each cell of kind \( i \) has capacity \( k_i - 1 \) \( (i = 1, \ldots, s) \). A generalization of the Pascal triangles of order \( k \) is also discussed.

1. INTRODUCTION

Denote by \( N_k(m, n) \) the number of distinct allocations of \( n \) indistinguishable objects into \( m \) distinguishable cells, each of which has capacity \( k - 1 \). It is well-known (see, e.g. Freund [6], Riordan [14, p. 104], and Bondarenko [3, p. 22], that

\[
N_k(m, n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n - kj + m - 1)}{m - 1},
\]

(1.1)

\[
N_k(m, n) = \sum_{j=0}^{k-1} N_k(m - 1, n - j),
\]

(1.2)

\[
N_k(m, n) = N_k(m, n - 1) + N_k(m - 1, n) - N_k(m - 1, n - k),
\]

(1.3)

and

\[
N_k(m, n) = \sum_{j=0}^{m} \binom{m}{j} N_{k-1}(j, n - j).
\]

(1.4)
Throughout the paper, for \( m, n \) integers, the binomial coefficient \( \binom{m}{n} \) is equal to 1, if \( m \geq 0 \) and \( n = 0 \) or \( m = n \); it is equal to \( \prod_{j=1}^{n}(m - j + 1)/\prod_{j=1}^{n}j \), if \( m > n > 0 \), and equals 0, otherwise.

The number \( N_k(m, n) \) has been used extensively in reliability and probability studies (see, e.g. Derman, Lieberman and Ross [5], Sen, Agarwal and Bhattacharya [15], Makri and Philippou [7], and Makri, Philippou and Psillakis [9]. Instead of \( N_k(m, n) \), some authors (e.g. Bondarenko [3], R. L. Ollerton and A. G. Shannon [11, 12] use the notation \( \binom{m}{n}^k \), and name the latter generalized binomial coefficient of order \( k \). For \( k = 2 \), relations (1.1) and (1.2) reduce to:

\[
N_2(m, n) = \binom{m}{n}^2 = \binom{m}{n}, \quad \text{and} \quad \binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}.
\]

As Freund [6] observed, recurrence (1.2), defines a generalized Pascal triangle as an array whose \( (m, n) \) entry \( (N_k(m, n)) \) equals the sum of the \( k \) entries above it and to the left \( (\sum_{j=0}^{k-1} N_k(m-1, n-j)) \). For more on generalized Pascal triangles, or to be more precise Pascal triangles of order \( k \), we refer to Philippou and Georghiou [13], Bollinger [1, 2], and Ollerton and Shannon [10].

In the present paper we generalize relations (1.1)-(1.4) to the case of \( s \) kinds of cells. This we do in Section 2. We also discuss, in Section 3, the corresponding generalized Pascal triangles.

### 2. RESTRICTED OCCUPANCY OF S KINDS OF CELLS

Presently we relax the assumption that each of the \( m \) cells has capacity \( k - 1 \) by assuming instead that there are \( s \) kinds of cells and each one of kind \( i \) has capacity \( k_i - 1 \) \((i = 1, \ldots, s)\). We first derive the following generalization of (1.1).

**Proposition 2.1:** For \( k = (k_1, \ldots, k_s) \) and \( m = (m_1, \ldots, m_s) \), denote by \( N_k(m, n) \) the number of distinct allocations of \( n \) indistinguishable objects into \( m \) distinguishable cells. Assume that each of \( m_i \) specified cells has capacity \( k_i - 1 \) \((i = 1, \ldots, s)\) and set \( m = m_1 + \ldots + m_s \).

Then,

\[
N_k(m, n) = \sum_{j_1=0}^{m_1} \ldots \sum_{j_s=0}^{m_s} (-1)^{j_1+\ldots+j_s} \binom{m_1}{j_1} \ldots \binom{m_s}{j_s} \left( \frac{m-1+n-k_1j_1-\ldots-k_sj_s}{m-1} \right). \tag{2.1}
\]

**Proof:** Let \( g(t) \) be the generating function of \( N_k(m, n) \). Then,
RESTRICTED OCCUPANCY OF S KINDS OF CELLS AND GENERALIZED PASCAL TRIANGLES

\[ g(t) = \sum_{n=0}^{\infty} N_k(m, n) t^n = \prod_{i=1}^{s} (1 + t + t^2 + \ldots + t^{k_i - 1})^{m_i} \]

\[ = \left[ \prod_{i=1}^{s} (1 - t^{k_i})^{m_i} \right] (1 - t)^{-m}, \quad m = \sum_{i=1}^{s} m_i \]

\[ = \left[ \prod_{i=1}^{s} \sum_{j_i=0}^{m_i} (-1)^{j_i} \binom{m_i}{j_i} t^{k_i j_i} \right] \sum_{j=0}^{\infty} \binom{m - 1 + j}{m - 1} t^j, \]

by the binomial theorem,

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \prod_{i=1}^{s} (-1)^{j_i} \binom{m_i}{j_i} \right] \binom{m - 1 + j}{m - 1} t^n, \]

where the inner summation is over all nonnegative integers \( j, j_1, j_2, \ldots, j_s \), satisfying the conditions \( j_i \leq m_i \quad (i = 1, \ldots, s) \) and \( j + \sum_{i=1}^{s} k_i j_i = n \). Therefore,

\[ N_k(m, n) = \sum \left[ \prod_{i=1}^{s} (-1)^{j_i} \binom{m_i}{j_i} \right] \binom{m - 1 + j}{m - 1}, \]

from which the proposition follows. \( \square \)

For \( s = 1 \), Proposition 1.1 reduces to relation (1.1). For \( s = 2 \), it reduces to

\[ N_{k_1, k_2}(m_1, m_2, n) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} (-1)^{j_1 + j_2} \binom{m_1}{j_1} \binom{m_2}{j_2} \binom{m - 1 + n - k_1 j_1 - k_2 j_2}{m - 1}, \quad (2.2) \]

a result derived and employed by Makri, Philippou and Psillakis [8] (2007a) to study Polya, inverse Polya and circular Polya distributions of order \( k \) for \( l \)-overlapping success runs. We proceed now to generalize recurrences (1.2) - (1.4).

**Proposition 2.2**: Let \( N_k(m, n) \) be as in Proposition 2.1. Then,

\[ N_k(m, n) = \sum_{j_1=0}^{k_1 - 1} N_k(m_1 - 1, m_2, \ldots, m_s, n - j_1), \quad (2.3) \]

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\[ N_k(m, n) = \sum_{j_i=0}^{k_i-1} N_k(m_1, \ldots, m_{i-1}, m_i-1, m_{i+1}, \ldots, m_s, n-j_i), \quad i = 2, \ldots, s-1, \quad (2.4) \]

and

\[ N_k(m, n) = \sum_{j_s=0}^{k_s-1} N_k(m_1, m_2, \ldots, m_{s-1}, m_s-1, n-j_s). \quad (2.5) \]

**Proof:** It suffices to show (2.3). We first note that by employing (2.1) and the Pascal triangle identity \[ \binom{m_1}{j_1} = \binom{m_1-1}{j_1-1} + \binom{m_1-1}{j_1} \], we get

\[ N_k(m, n) = S_1 + S_2 \quad (2.6) \]

with

\[ S_1 = \sum_{j_1=1, j_2=0}^{m_1} \sum_{j_s=0}^{m_s} \ldots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1} \binom{m_2}{j_2} \ldots \binom{m_s}{j_s} (m - 1 + n - \sum_{i=1}^{s} k_i j_i) \]

\[ = \sum_{j_1'=0, j_2=0}^{m_1-1} \sum_{j_2=0}^{m_2} \ldots \sum_{j_s=0}^{m_s} (-1)^{j_1'+j_2+\ldots+j_s+1} \binom{m_1-1}{j_1'} \binom{m_2}{j_2} \ldots \binom{m_s}{j_s} \]
\[ \times \left( m - 1 - k_1 + n - k_1 j_1' - \sum_{i=2}^{s} k_i j_i \right) \quad \frac{m-1}{m-1} \]

on setting \( j_1' = j_1 - 1 \), and

\[ S_2 = \sum_{j_1=0, j_2=0}^{m_1-1} \sum_{j_s=0}^{m_s} \ldots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1} \binom{m_2}{j_2} \ldots \binom{m_s}{j_s} (m - 1 + n - \sum_{i=1}^{s} k_i j_i) \]

\[ = \sum_{j_1=0, j_2=0}^{m_1-1} \sum_{j_2=0}^{m_2} \ldots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1} \binom{m_2}{j_2} \ldots \binom{m_s}{j_s} \]
\[ \times \left[ (m - 1 - k_1 + n - \sum_{i=1}^{s} k_i j_i) \frac{k_1-1}{m-1} + \sum_{j=0}^{k_1-1} \left( \frac{(m-1)-1+n-j - \sum_{i=1}^{s} k_i j_i}{(m-1)-1} \right) \right]. \]
The last equality follows by means of the “vertical” recurrence relation (Charalambides [4, p. 129]

\[
\binom{x}{k} = \binom{x-r-1}{k} + \sum_{j=0}^{r} \binom{x-j-1}{k-1},
\]

which holds true for any real number \(x\) and any nonnegative integer \(k\). By interchanging the order of summation we obtain that

\[
S_2 = \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \binom{m - 1 - k_1 + n - \sum_{i=1}^{s} k_i j_i}{m-1},
\]

\[
+ \sum_{j=0}^{k_1-1} \left[ \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \right] \binom{m - 1 - 1 + n - j - \sum_{i=1}^{s} k_i j_i}{m-1 - 1},
\]

\[
= -S_1 + \sum_{j=0}^{k_1-1} N_k(m_1 - 1, m_2, \ldots, m_s, n - j).
\]

Substituting \(S_2\) in (2.6) the proposition follows.

For \(s = 1\) Proposition 2.2 reduces to (1.2). For \(s = 2\), it reduces to

\[
N_{k_1,k_2}(m_1, m_2, n) = \sum_{j_1=0}^{k_1-1} N_{k_1,k_2}(m_1 - 1, m_2, n - j_1),
\]

(2.7)

and

\[
N_{k_1,k_2}(m_1, m_2, n) = \sum_{j_2=0}^{k_2-1} N_{k_1,k_2}(m_1, m_2 - 1, n - j_2).
\]

(2.8)

Furthermore, by usage of (2.3)-(2.5) we get

\[
N_k(m, n) = \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_s=0}^{k_s-1} N_k(m_1 - 1, \ldots, m_s - 1, n - j_1 - \ldots - j_s),
\]

(2.9)
which, for \( s = 2 \), reduces to

\[
N_{k_1,k_2}(m_1,m_2,n) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} N_{k_1,k_2}(m_1 - 1,m_2 - 1,n - j_1 - j_2).
\]

**Proposition 2.3:** Let \( N_k(m,n) \) be as in Proposition 2.1. Then,

\[
N_k(m,n) = N_k(m,n-1) + N_k(m_1-1,m_2,...,m_s,n) - N_k(m_1-1,m_2,...,m_s,n-k_1),
\]

\[\tag{2.11}\]

\[
N_k(m,n) = N_k(m,n-1) + N_k(m_1,...,m_{i-1},m_i-1,m_{i+1},...,m_s,n)
- N_k(m_1,...,m_{i-1},m_i-1,m_{i+1},...,m_s,n-k_i), \quad i = 2,...,s-1,
\]

\[\tag{2.12}\]

and

\[
N_k(m,n) = N_k(m,n-1) + N_k(m_1,m_2,...,m_s-1,n) - N_k(m_1,m_2,...,m_s-1,m_s-1,n-k_s).
\]

\[\tag{2.13}\]

**Proof:** It suffices to show (2.11). By Proposition 2.1 and the Pascal triangle identity

\[
N_k(m,n) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\cdots+j_s} \left( \binom{m_1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \right) \left( m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i \right) m + 1
\]

\[+ \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\cdots+j_s} \left( \binom{m_1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \right) \left( m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i \right) m + 1
\]

\[= N_k(m,n-1)
\]

\[+ \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\cdots+j_s} \left( \binom{m_1-1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \right) \left( m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i \right) m + 1
\]

\[+ \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\cdots+j_s} \left( \binom{m_1-1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \right) \left( m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i \right) m + 1
\]

\[= N_k(m,n-1) + N_k(m_1-1,m_2,...,m_s,n)
\]

\[+ \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\cdots+j_s} \left( \binom{m_1-1}{j_1} \binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \right) \left( m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i \right) m + 1
\]
The result follows by setting $j_1 - 1 = j'_1$ in the sum of the last equality. □

For $s = 1$ Proposition 2.3 reduces to (1.3). For $s = 2$, it reduces to

$$N_{k_1,k_2}(m_1,m_2,n) = N_{k_1,k_2}(m_1,m_2,n-1) + N_{k_1,k_2}(m_1-1,m_2,n) - N_{k_1,k_2}(m_1-1,m_2,n-k_1),$$

and

$$N_{k_1,k_2}(m_1,m_2,n) = N_{k_1,k_2}(m_1,m_2,n-1) + N_{k_1,k_2}(m_1,m_2-1,n) - N_{k_1,k_2}(m_1,m_2-1,n-k_2).$$

\(2.14\)
\(2.15\)

**Proposition 2.4:** Let $N_k(m,n)$ be as in Proposition 2.1. Then,

$$N_k(m,n) = \sum_{j_1=0}^{m_1} \cdots \sum_{j_s=0}^{m_s} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} N_{k-1}(j_1, \ldots, j_s, n - j_1 - \cdots - j_s).$$

\(2.16\)

**Proof:** We consider the proof of (2.16) as a classical occupancy problem. Let $A$ be the set of allocations of $n$ indistinguishable objects into $m$ distinguishable cells such that each of $m_i$ specified cells may be occupied by at most $k_i - 1$ objects (cells of the $i$th kind), $i = 1, \ldots, s$ ($m = m_1 + \cdots + m_s$).

For $i = 1, \ldots, s$, let $A_{j_i}^{(i)}$ be the subset of these allocations in which $j_i$ cells, $j_i = 0, 1, \ldots, m_i$, of the $i$th kind are occupied (and consequently the remaining $m_i - j_i$ cells of the $i$th kind remain empty). For given $j_1, \ldots, j_s$ and any specified selection of $j_1$ cells out of $m_1$ of the 1st kind, $\ldots, j_s$ cells out of $m_s$ of the $s$th kind, one object is placed in each of these $j_1 + \cdots + j_s$ specified cells. Next, note that the number of allocations of the remaining $n - (j_1 + \cdots + j_s)$ objects into the $j_1 + \cdots + j_s$ cells, under the restrictions of the capacities of the cells, equals

$$N_{k-1}(j_1, \ldots, j_s, n - (j_1 + \cdots + j_s))$$

by Proposition 2.1. Further, the $j_1, \ldots, j_s$ cells can be chosen in

$$\binom{m_1}{j_1} \cdots \binom{m_s}{j_s}, \quad j_i = 0, 1, \ldots, m_i, \quad i = 1, 2, \ldots, s$$

ways. So, according to the multiplicative principle, the number of the elements of the set $A_{j_1}^{(1)} \cap \cdots \cap A_{j_s}^{(s)}$ equals

$$\binom{m_1}{j_1} \cdots \binom{m_s}{j_s} N_{k-1}(j_1, \ldots, j_s, n - (j_1 + \cdots + j_s)).$$

Thus, summing for all values of $j_i = 0, 1, \ldots, m_i$, $i = 1, \ldots, s$, according to the addition principle, we deduce (2.16). □
For $s = 1$ Proposition 2.4 reduces to (1.4). For $s = 2$, it reduces to

$$N_{k_1,k_2}(m_1,m_2,n) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \binom{m_1}{j_1} \binom{m_2}{j_2} N_{k_1-1,k_2-1}(j_1,j_2,n-j_1-j_2).$$

(2.17)

3. GENERALIZED PASCAL TRIANGLES OF ORDER $k$

In this section, we note that the $s$ recurrences (2.3)-(2.5) define a generalized Pascal triangle (hyper cube), which we call Pascal triangle of order $k$ and denote by $T_k(m,n)$, as the hyper cube whose $(m,n)$ entry $N_k(m,n)$ equals any one of the $k_i$ sums ($i = 1,\ldots,s$) appearing on the right-hand side of (2.3)-(2.5). For example, recurrence (2.3) gives the $(m_1,n)$ entry $N_{k_1}(m_1,n)$ of $T_{k_1}(m_1,n)$ as the sum of the $k_1$ entries $N_k(m_1-1,m_2,\ldots,m_s,n-j), j = 0,1,\ldots,k_1-1$. For $s = 2$, the $(m_1,m_2,n)$ entry of the Pascal triangle (cube) of order $(k_1,k_2)$ equals the sum of the $k_1$ entries $N_{k_1,k_2}(m_1-1,m_2,n-j), j = 0,1,\ldots,k_1-1$. It is also equal to the sum of the $k_2$ entries $N_{k_1,k_2}(m_1,m_2-1,n-j), j = 0,1,\ldots,k_2-1$.

Geometrically, we could use recurrence (2.7) to construct a cube with entries $N_{k_1,k_2}(m_1,m_2,n)$. Consider a cube such that, on its upper (horizontal) side ($P_u$), a generalized Pascal triangle of order $k_1$, $T_{k_1}(m_1,n)$ is created Freund [6], e.g., its first row $m_1 = 0$ consists of a 1 and no other entries and each other entry is obtained as the sum of the entry immediately above and the $k_1-1$ entries to its left.

Next, on the left vertical side of the cube ($P_v$), perpendicular to the upper side, a generalized Pascal triangle of order $k_2$, $T_{k_2}(m_2,n)$ is created (see the following figure, which provides an illustration for $k_1 = 3$, $k_2 = 4$).

Note that the $(m_1,n)$ entry of $T_{k_1}(m_1,n)$ is simultaneously the $(m_1,0,n)$ entry of the cube, and the $(m_2,n)$ entry of $T_{k_2}(m_2,n)$ is simultaneously the $(0,m_2,n)$ entry of the cube.

For a given value of $m_2 = m$ we consider a plane parallel to the upper side of the cube which intersects the left vertical side of the cube at the row $m_2 = m$ of $T_{k_2}(m_2,n)$. On this new
plane an array is constructed with its first row ($m_1 = 0$) being the $m_2 = m$ row of $T_{k_2}(m_2, n)$ and each other entry is obtained as the sum of the entry immediately above and the $k_1 - 1$ entries to its left. $N_{k_1,k_2}(m_1,m_2,n)$, which represents the number of distinct allocations of $n$ indistinguishable objects into $m_1$ distinguishable cells each of which has capacity $k_1 - 1$ and $m_2$ distinguishable cells each of which has capacity $k_2 - 1$, is the $(m_1,n)$ entry of this array. A similar procedure could be followed using recurrence (2.8).

To make it more clear, we note that in order to calculate $N_{k_1,k_2}(u,v,n)$ we first construct $T_{k_2}(m_2,n)$ until its line $m_2 = v$. In the sequel we construct an array $(a_{m_1,n})$ with its first row ($m_1 = 0$) being the $m_2 = v$ row of $T_{k_2}(m_2,n)$ and each other entry of the array is obtained as the sum of the entry above and $k_1 - 1$ entries to the left of the one immediately above.

As an example, we give the calculation of $N_{3,4}(m_1,6,n)$. First, we construct $T_{4}(m_2,n)$.

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<th>2</th>
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Then we construct $T_{3,4}(m_1,6,n) = T_{3}(m_1,n)$ with $N_{3}(0,n) = N_{4}(6,n)$,

<table>
<thead>
<tr>
<th>$m_1 \setminus n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>6</td>
<td>21</td>
<td>56</td>
<td>120</td>
<td>216</td>
<td>336</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>7</td>
<td>28</td>
<td>83</td>
<td>197</td>
<td>392</td>
<td>672</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
<td>36</td>
<td>118</td>
<td>308</td>
<td>672</td>
<td>1261</td>
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<tr>
<td>3</td>
<td>1</td>
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<td>45</td>
<td>162</td>
<td>462</td>
<td>1098</td>
<td>2241</td>
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<tr>
<td>4</td>
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<td>10</td>
<td>55</td>
<td>216</td>
<td>669</td>
<td>1722</td>
<td>2865</td>
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<tr>
<td>5</td>
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<td>66</td>
<td>281</td>
<td>940</td>
<td>2607</td>
<td>3750</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>12</td>
<td>78</td>
<td>358</td>
<td>1287</td>
<td>3828</td>
<td>4971</td>
</tr>
</tbody>
</table>

from which $N_{3,4}(m_1,6,n)$ are readily available. For example,

$N_{3,4}(2,6,5) = N_{3}(2,5) = 672,$

$N_{3,4}(5,6,3) = N_{3}(5,3) = 281,$

$N_{3,4}(6,6,4) = N_{3}(6,4) = 1287.$
REFERENCES


AMS Classification Numbers: 05A10, 11B65