A COMBINATORIAL APPROACH TO FIBONOMIAL COEFFICIENTS

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Abstract. A combinatorial argument is used to explain the integrality of Fibonomial coefficients and their generalizations. The numerator of the Fibonomial coefficient counts tilings of staggered lengths, which can be decomposed into a sum of integers, such that each integer is a multiple of the denominator of the Fibonomial coefficient. By colorizing this argument, we can extend this result from Fibonacci numbers to arbitrary Lucas sequences.

1. Introduction

The Fibonomial Coefficient \( \binom{n}{k}_F \) is defined, for \( 0 < k \leq n \), by replacing each integer appearing in the numerator and denominator of \( \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots1} \) with its respective Fibonacci number. That is,

\[
\binom{n}{k}_F = \frac{F_nF_{n-1}\cdots F_{n-k+1}}{F_kF_{k-1}\cdots F_1}.
\]

For example, \( \binom{7}{3}_F = \frac{F_7F_6F_5F_3F_2F_1}{F_kF_{k-1}\cdots F_1} = 13 \cdot 8 \cdot 5 \cdot 2 \cdot 1 = 1340 \).

It is, at first, surprising that this quantity will always take on integer values. This can be shown by an induction argument by replacing \( F_n \) in the numerator with \( F_kF_{n-k+1} + F_{k-1}F_{n-k} \), resulting in

\[
\binom{n}{k}_F = \frac{F_nF_{n-1}\cdots F_{n-k+1}}{F_kF_{k-1}\cdots F_1}.
\]

By similar reasoning, this integrality property holds for any Lucas sequence defined by \( U_0 = 0, U_1 = a \) and for \( n \geq 2, U_n = aU_{n-1} + bU_{n-2} \), and we define

\[
\binom{n}{k}_U = \frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_kU_{k-1}\cdots U_1}.
\]

In this note, we combinatorially explain the integrality of \( \binom{n}{k}_F \) and \( \binom{n}{k}_U \) by a tiling interpretation, answering a question proposed in Benjamin and Quinn’s book, *Proofs That Really Count* [1].

2. Staggered Tilings

It is well-known that for \( n \geq 0 \), \( f_n = F_{n+1} \) counts tilings of a \( 1 \times n \) board with squares and dominoes [1]. For example, \( f_4 = 5 \) counts the five tilings of length four, where \( s \) denotes a square tile and \( d \) denotes and domino tile: \( sss, ssd, sds, dss, dd \). Hence, for \( \binom{n}{k}_F = \frac{f_{n-1}f_{n-2}\cdots f_{n-k}}{f_{k-1}f_{k-2}\cdots f_0} \), the numerator counts the ways to simultaneously tile boards of length \( n-1, n-2, \ldots, n-k \). The challenge is to find disjoint “subtilings” of lengths \( k-1, k-2, \ldots, 0 \) that can be described in a precise way. Suppose \( T_1, T_2, \ldots, T_k \) are tilings with respective lengths \( n-1, n-2, \ldots, n-k \). We begin by looking for a tiling of length \( k-1 \).
THE FIBONACCI QUARTERLY

If $T_1$ is “breakable” at cell $k - 1$, which can happen $f_{k-1}f_{n-k}$ ways, then we have found a tiling of length $k - 1$. We would then look for a tiling of length $k - 2$, starting with tiling $T_2$.

Otherwise, $T_1$ is breakable at cell $k - 2$, followed by a domino (which happens $f_{k-2}f_{n-k-1}$ ways). Here, we “throw away” cells 1 through $n$, which we call $T_{k+1}$. (Note that $T_{k+1}$ has length $n - k - 1$, which is one less than the length of $T_k$.) We would then continue our search for a tiling of length $k - 1$ in $T_2$, then $T_3$, and so on, creating $T_{k+2}, T_{k+3}$, and so on as we go, until we eventually find a tiling $T_{x_1}$ that is breakable at cell $k - 1$. (We are guaranteed that $x_1 \leq n - k + 1$ since $T_{n-k+1}$ has length $k - 1$.) At this point, we disregard everything in $T_{x_1}$ and look for a tiling of length $k - 2$, beginning with tiling $T_{x_1+1}$.

Following this procedure, we have, for $1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n$, the number of tilings $T_1, T_2, \ldots, T_k$ that lead to finding a tiling of length $k - i$ at the beginning of tiling $T_{x_i}$ is

$$f_{x_i-1} f_{k-2} f_{n-x_i-1-(k-1)} f_{x_i-1} \cdots f_0  
= f_{n-1} f_{n-2} \cdots f_{n-k}$$

$$= f_{k-1} f_{k-2} f_{k-3} \cdots f_1 \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} (f_{k-1-i})^{x_i-x_i-1} f_{n-x_i-(k-i)}$$

$$= F_k F_{k-1} F_{k-2} \cdots F_2 F_1 \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} (F_{k-i})^{x_i-x_i-1} F_{n-x_i-(k-i)+1}.$$ 

That is,

$$\binom{n}{k}_F = \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} F_{k-i}^{x_i-x_i-1} F_{n-x_i-(k-i)+1}.$$ 

This theorem has a natural Lucas sequence generalization. For positive integers $a, b$, it is shown in [1] that $u_n = U_{n+1}$ counts colored tilings of length $n$, where there are $a$ colors of squares and $b$ colors of dominoes. (More generally, if $a$ and $b$ are any complex numbers, $u_n$ counts the total weight of length $n$ tilings, where squares and dominoes have respective weights $a$ and $b$, and the weight of a tiling is the product of the weights of its tiles.) By virtually the same argument as before, we have

$$\binom{n}{k}_U = \sum_{1 \leq x_1 < x_2 < \cdots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} b^{x_i-x_i-1} U_{k-i}^{x_i-x_i-1} U_{n-x_i-(k-i)+1}.$$ 

The presence of the $b^{x_k-1-(k-1)}$ term accounts for the $x_{k-1} - (k - 1)$ dominoes that caused $x_{k-1} - (k - 1)$ tilings to be unbreakable at their desired spot.

As an immediate corollary, we note that the right hand side of this identity is a multiple of $b$, unless $x_i = i$ for $i = 1, 2, \ldots, k - 1$. It follows that

$$\binom{n}{k}_U \equiv U_{n-k+1}^{k-1} \pmod{b}.$$ 

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REFERENCES


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