ON HOFSTADTER’S MARRIED FUNCTIONS

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Abstract. In this note we show that Hofstadter’s married functions generated by the intertwined system of recurrences
a(0) = 1, b(0) = 0, b(n) = n − a(b(n − 1)), a(n) = n − b(a(n − 1)) has the solutions
a(n) = ⌊(n + 1)φ−1⌋ + ε1(n) and b(n) = ⌊(n + 1)φ−1⌋ − ε2(n), where φ is the golden ratio and ε1, ε2 are indicator functions of Fibonacci numbers diminished by 1.

1. Introduction

In his well-known book “Gödel, Escher, Bach: An Eternal Golden Braid,” D. R. Hofstadter [9] introduces several recurrences which give rise to particularly intriguing integer sequences. Mention, for instance, the famous Hofstadter’s Q–sequence (also known as Meta-Fibonacci sequence [6], entry A005185 in Sloane’s Encyclopedia [13]) which is defined by
Q(1) = Q(2) = 1 and
Q(n) = Q(n − Q(n − 1)) + Q(n − Q(n − 2)) for n > 2. (1.1)

Each term of the sequence is the sum of two preceding terms but, in contrast to the Fibonacci sequence, not necessarily the two last terms. The sequence Q(n) shows an erratic behavior as well as a certain degree of regularity (see [12]). Another link to the sequence of Fibonacci numbers (Fk)k≥1 = 1, 1, 2, 3, 5, 8, 13 . . . is given in Hofstadter’s G–sequence (A005206), which is generated by G(0) = 0 and
G(n) = n − G(G(n − 1)) for n > 0. (1.2)

It has been shown [11] that if n = Fr(1) + Fr(2) + · · · + Fr(m) is the Zeckendorf expansion of n, then G(n) = Fr(1)−1 + Fr(2)−1 + · · · + Fr(m)−1. Furthermore, Downey/Griswold [1] and Granville/Rasson [3] proved an explicit formula for G(n), namely,
G(n) = ⌊(n + 1)µ⌋, (1.3)
where µ = (√5 − 1)/2 = φ−1 and φ = (√5 + 1)/2 is the golden ratio. Various generalizations of (1.2) have been investigated in [1, 10].

In this note we are concerned with Hofstadter’s “married functions” (or Hofstadter’s male–female sequences [14]) defined by two intertwined functions a(n) and b(n) (see p.137 of [9]).

Definition 1.1. Denote by a(n) and b(n) the sequences defined by a(0) = 1, b(0) = 0 and for n > 0 by the intertwined system of recurrences
\[
\begin{align*}
  b(n) &= n - a(b(n - 1)), \\
  a(n) &= n - b(a(n - 1)).
\end{align*}
\]

The first one hundred terms of the “female” sequence a(n) (A005378) and, respectively, of the “male” sequence b(n) (A005379) are given in Table 1.
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Table 1. Values of \(a(n), b(n)\) for \(0 \leq n \leq 99\)

<table>
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<th>n</th>
<th>(a(n))</th>
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A simple inductive argument shows that \(0 < a(n) \leq n+1\) and \(0 \leq b(n) \leq n\), thus ensuring that both sequences are well-defined for all \(n \geq 0\) by the recursion (1.4). J. Grytczuk [4, 5] provided a general framework for the recursions (1.2) and (1.4), namely by linking them to Richelieu cryptosystems and symbolic dynamics on infinite words. We here prove explicit formulas for \(a(n)\) and \(b(n)\) which, somehow as a surprise, involve both the quantity \(\mu\) from formula (1.3) and the explicit notion of Fibonacci numbers \(F_k\).

Our main result is the following theorem.

**Theorem 1.2.** For all \(n \geq 0\) there hold

\[
\begin{align*}
a(n) &= \lfloor (n + 1)\mu \rfloor + \varepsilon_1(n), \\
b(n) &= \lfloor (n + 1)\mu \rfloor - \varepsilon_2(n),
\end{align*}
\]

where for \(k \geq 1,\)

\[
\varepsilon_1(n) = \begin{cases} 
1, & \text{if } n = F_{2k} - 1; \\
0, & \text{else.}
\end{cases} \quad \varepsilon_2(n) = \begin{cases} 
1, & \text{if } n = F_{2k+1} - 1; \\
0, & \text{else.}
\end{cases}
\]
2. Proof

The proof of Theorem 1.2 involves an inductive argument, where several cases have to be taken into account. We begin with two lemmas which link the floor expression of the statement (a so-called Beatty sequence) with the Fibonacci numbers.

Lemma 2.1. Let \( n \geq 4 \). Then

1. \( \lfloor n \mu \rfloor = F_{2k} - 1 \) if and only if \( n = F_{2k+1} - 1 \).
2. \( \lfloor n \mu \rfloor = F_{2k-1} - 1 \) if and only if \( n = F_{2k} - 1 \) or \( n = F_{2k} + 1 \).
3. \( \lfloor n \mu \rfloor = F_{2k} \) if and only if \( n = F_{2k+1} \) or \( n = F_{2k+1} + 1 \).
4. \( \lfloor n \mu \rfloor = F_{2k-1} \) if and only if \( n = F_{2k} + 1 \).

Proof. The golden ratio \( \phi = 1.6180339 \ldots \) has the simple continued fraction expansion \([1, 1, 1, \ldots]\). Thus, for any \( k \geq 2 \), the quotient

\[
\frac{F_k}{F_{k-1}} = 1 + \frac{1}{\frac{F_{k-1}}{F_{k-2}}} = \ldots = \left[1, 1, \ldots, \frac{F_2}{F_1}\right] = [1, 1, 1, \ldots, 1]
\]

is just the \((k - 1)\)th convergent of \( \phi \). Consequently, general theory of continued fractions (e.g., see Theorem 163 and Theorem 171 in [8]) yields

\[
\phi - \frac{F_k}{F_{k-1}} = \frac{(-1)^k}{F_{k-1}(\phi F_{k-1} + F_{k-2})}.
\]

Since \( \phi F_{k-1} + F_{k-2} \leq (\phi + 1)F_{k-1} < 3F_{k-1} \) and \( \phi F_{k-1} + F_{k-2} > F_{k-1} \), we have the estimates

\[
\frac{1}{3F_{2k-1}^2} < \phi - \frac{F_{2k}}{F_{2k-1}} < \frac{1}{F_{2k-1}^2}
\]

and

\[
\frac{1}{3F_{2k}^2} < \frac{F_{2k+1}}{F_{2k}} - \phi < \frac{1}{F_{2k}^2}
\]

(2.1)

First, consider case (1). Obviously, the left hand equation is equivalent to \( F_{2k} - 1 \leq n \phi^{-1} < F_{2k} \) for \( k \geq 2 \), which again can be rewritten as

\[
0 < \phi F_{2k} - n \leq \phi.
\]

(2.3)

We have to prove that this inequality holds if and only if \( n = F_{2k+1} - 1 \). Instead of directly proving (1), we show the inequality

\[
\phi - 1 < \phi F_{2k} - (F_{2k+1} - 1) \leq 1,
\]

(2.4)

which implies (1). Namely, since \( 0 < \phi - 1 < 1 < \phi \) and (2.3), this shows the “\( \leq \)” part of (1). On the other hand, the bounds in (2.3) ensure that \( \phi F_{2k} - F_{2k+1} - 2 \leq 0 \) and \( \phi F_{2k} - (F_{2k+1} - 2) > \phi \), so that by linearity there are no other values of \( n \) satisfying (2.3). This gives the “\( \Rightarrow \)” part of (1). Now, as for the proof of (2.4), we see that by (2.2),

\[
\phi - 1 < 1 - \frac{1}{F_{2k}} < 1 + F_{2k} \left( \phi - \frac{F_{2k+1}}{F_{2k}} \right) < 1 - \frac{1}{3F_{2k}} < 1,
\]

where the first and last inequality holds by trivial means and the middle term corresponds to the middle term in (2.4). This finishes the proof of (1).

Similarly, in case (2) we have

\[
0 < \phi F_{2k-1} - n \leq \phi
\]

(2.5)
for \( k \geq 3 \) and it is sufficient to show that \( 0 < \phi F_{2k-1} - F_{2k} \leq \phi - 1 \), since then only \( n = F_{2k-1} \) and \( n = F_{2k} \) can satisfy (2.5). Here, (2.1) yields
\[
0 < \frac{1}{3F_{2k-1}} < F_{2k-1} \left( \phi - \frac{F_{2k}}{F_{2k-1}} \right) < \frac{1}{F_{2k-1}} < \phi - 1,
\]
which gives the equivalence in case (2).

For case (3) it suffices to ensure that \( 0 \leq F_{2k+1} - \phi F_{2k} < \phi - 1 \) for \( k \geq 2 \), which holds true due to
\[
0 < \frac{1}{3F_{2k}} < F_{2k} \left( \frac{F_{2k+1}}{F_{2k}} - \phi \right) < \frac{1}{F_{2k}} < \phi - 1.
\]
Finally, in case (4) the condition \( \phi - 1 \leq F_{2k} + 1 - \phi F_{2k-1} < 1 \) is guaranteed for \( k \geq 3 \) by
\[
\phi - 1 < 1 - \frac{1}{F_{2k-1}} < 1 + F_{2k+1} \left( \frac{F_{2k}}{F_{2k-1}} - \phi \right) < 1 - \frac{1}{3F_{2k-1}} < 1
\]
and by directly checking (4) for \( k = 2 \).

In order to get our inductive argument to work later, we apply Lemma 2.1 to evaluate the indicator functions \( \varepsilon_1, \varepsilon_2 \) at \( \lfloor n\mu \rfloor \). Obviously, by definition,
\[
\varepsilon_1(0) = \varepsilon_2(1) = 1, \quad \varepsilon_1(1) = \varepsilon_2(0) = 0.
\]
Moreover, it is clear from Lemma 2.1, cases (1) and (2), that for all \( n \geq 0 \) there hold
\[
\varepsilon_1(\lfloor n\mu \rfloor) = \begin{cases} 1, & \text{if } n = F_{2k+1} - 1; \\ 0, & \text{else}. \end{cases}
\]
\[
\varepsilon_2(\lfloor n\mu \rfloor) = \begin{cases} 1, & \text{if } n = F_{2k} - 1 \text{ or } n = F_{2k}; \\ 0, & \text{else}. \end{cases}
\]

Our second lemma concerns nested Beatty sequences of type \( \lfloor \mu \lfloor \mu \mu \rfloor + C \rfloor \) and will be useful to get rid of subsequent floors at specified points in the induction step.

**Lemma 2.2.**

(1) For all \( n \geq 0 \) holds \( \lfloor \mu n + \mu \rfloor = n - \lfloor \mu \lfloor \mu n \rfloor + \mu \rfloor \).

(2) For \( n = F_{2k} \) with \( k \geq 2 \) holds \( \lfloor \mu n + \mu \rfloor = n - \lfloor \mu \lfloor \mu n \rfloor \rfloor + 2\mu \).

(3) For \( n = F_{2k+1} \) with \( k \geq 2 \) holds \( \lfloor \mu n + \mu \rfloor = n - \lfloor \mu \lfloor \mu n \rfloor \rfloor - 1 \).

**Proof.** Equation (1) is Lemma 1 in [1]. We proceed with the proof of (2) and (3). First, let \( k \geq 3 \). Concerning statement (2), we observe with help of Lemma 2.1 (cases (2), (3), (4) therein), that
\[
F_{2k} - \lfloor \mu F_{2k} \rfloor + 2\mu = F_{2k} - \lfloor \mu (F_{2k-1} + 1) \rfloor = F_{2k} - F_{2k-2} = F_{2k-1} = \lfloor \mu F_{2k} \rfloor.
\]
Similarly, for statement (3) we use Lemma 2.1 (cases (3), (2), (3), respectively) and obtain
\[
F_{2k+1} - \lfloor \mu F_{2k+1} \rfloor - 1 = F_{2k+1} - \lfloor \mu F_{2k} \rfloor - 1 = F_{2k+1} - (F_{2k-1} - 1) - 1 = F_{2k} = \lfloor \mu (F_{2k+1} + 1) \rfloor.
\]
The case \( k = 2 \) is easily checked by hand. \( \Box \)

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We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. To begin with, we observe that the formulas given in Theorem 1.2 are true if \( n \in \{0, 1, 2\} \) (e.g., compare with Table 1). Suppose now \( n \geq 3 \). According to (1.4) we have to show that

\[
\lfloor (n+1)\mu \rfloor - \varepsilon_2(n) = n - \lfloor \mu (n\mu - \varepsilon_2(n-1) + 1) \rfloor
\]

and

\[
\lfloor (n+1)\mu \rfloor + \varepsilon_1(n) = n - \lfloor \mu (n\mu + \varepsilon_1(n-1) + 1) \rfloor
\]

We distinguish several cases on \( n \).

First, if \( n \neq F_{2k}, F_{2k} - 1, F_{2k+1} - 1 \) then by definition of the indicator functions,

\[\varepsilon_1(n) = \varepsilon_1(n-1) = \varepsilon_2(n) = \varepsilon_2(n-1) = 0.\]

Furthermore, by (2.7) and (2.8),

\[\varepsilon_1(\lfloor n\mu \rfloor) = \varepsilon_2(\lfloor n\mu \rfloor) = 0.\]

Then, the equalities (2.9) and (2.10) hold by (1) of Lemma 2.2.

If \( n = F_{2k+1} - 1 \), then

\[\varepsilon_1(n) = \varepsilon_1(n-1) = \varepsilon_2(n-1) = \varepsilon_2(\lfloor n\mu \rfloor) = 0 \quad \text{and} \quad \varepsilon_2(n) = \varepsilon_1(\lfloor n\mu \rfloor) = 1\]

and again (1) from Lemma 2.2 yields (2.9) and (2.10). In the same style, for \( n = F_{2k} - 1 \) we have

\[\varepsilon_1(n) = \varepsilon_2(\lfloor n\mu \rfloor) = 1 \quad \text{and} \quad \varepsilon_1(n-1) = \varepsilon_2(n) = \varepsilon_2(n-1) = \varepsilon_1(\lfloor n\mu \rfloor) = 0\]

and (2.9), (2.10) by (1) from Lemma 2.2. Moreover, for \( n = F_{2k} \) we have

\[\varepsilon_2(n) = \varepsilon_2(n-1) = \varepsilon_1(\lfloor n\mu \rfloor) = 0\]

and (2.9) again by (1). On the other hand, for \( n = F_{2k+1} \) it follows

\[\varepsilon_1(n) = \varepsilon_1(n-1) = \varepsilon_2(\lfloor n\mu \rfloor) = 0\]

and (2.10).

It remains to prove (2.9) for \( n = F_{2k+1} \) and (2.10) for \( n = F_{2k} \). For suppose \( n = F_{2k+1} \), then \( \varepsilon_2(n) = 0 \) and

\[\varepsilon_2(n-1) = \varepsilon_1(\lfloor n\mu \rfloor - 1) = 1.\]

Now, statement (3) of Lemma 2.2 gives equation (2.9). Finally, if \( n = F_{2k} \) then \( \varepsilon_1(n) = \varepsilon_2(\lfloor n\mu \rfloor + 1) = 0 \) and \( \varepsilon_1(n-1) = 1 \), and (2.10) corresponds to (2) of Lemma 2.2. This finishes the proof of Theorem 1.2.

\[\square\]
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3. Epilogue

It would be interesting to know whether there exist explicit formulas for (what we may call) Hofstadter’s parents-child sequences \(a(n), b(n)\) and \(c(n)\) defined by \(a(0) = 1, b(0) = c(0) = 0\) and

\[
\begin{align*}
    b(n) &= n - c(b(n - 1)), \\
    a(n) &= n - b(a(n - 1)), \\
    c(n) &= n - a(c(n - 1))
\end{align*}
\]

(3.1)

Consider, for instance, the “mother” sequence

\[a(n) = 1, 0, 2, 1, 3, 3, 4, 5, 7, 7, 8, 7, 10, 8, 10, \ldots\]

We are surprised by the fact that \(a(n) = \lceil (n + 1)\mu \rceil\) for \(4 \leq n \leq 8, 30 \leq n \leq 40, 140 \leq n \leq 176, 606 \leq n \leq 752\) etc., while on all other intervals irregular oscillations can be observed.

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References


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