RAMSEY RESULTS INVOLVING THE FIBONACCI NUMBERS

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Abstract. A collection $\mathcal{A}$ of sequences of positive integers is called *regular* if for all positive integers k and r, there is a least positive integer $n = n(k, r)$ such that for every partition of $\{1, 2, \ldots, n\}$ into r subsets, there is some subset that contains a k-term sequence belonging to $\mathcal{A}$. In this paper we examine the regularity of families related to the Fibonacci numbers. In particular, we consider the regularity of the family of arithmetic progressions whose gaps are Fibonacci numbers, the family of increasing sequences (not necessarily arithmetic progressions) whose gaps are Fibonacci numbers, and the family of all sequences satisfying the Fibonacci recurrence $x_i = x_{i-1} + x_{i-2}$.

1. Introduction

Many results in integer Ramsey theory take the following general form, where $\mathcal{A}$ is a given collection of sequences of integers and k and r are fixed positive integers.

> There is a least positive integer $n = n(\mathcal{A}, k; r)$ such that for every partition of $[1, n] = \{1, 2, \ldots, n\}$ into r subsets, at least one of the subsets must contain a k-term member of $\mathcal{A}$.

Two classical results of this nature are Schur's Theorem and van der Waerden's Theorem. Schur's Theorem [6] states that for all positive integers r, there is a least positive integer $s = s(r)$ such that whenever $[1, s]$ is partitioned into r subsets, at least one of the subsets will contain a triple (x, y, z) (not necessarily distinct) such that $x + y = z$. Van der Waerden's Theorem [7] states that for all positive integers k and r, there is a least positive integer $w = w(k; r)$ such that for every partition of $[1, w]$ into r subsets, at least one subset will contain a k-term arithmetic progression. For these two theorems, $\mathcal{A}$ is the family of solutions to $x + y = z$, and the family of arithmetic progressions, respectively.

The purpose of this note is to examine the Ramsey properties that result when $\mathcal{A}$ is a family of sequences defined in terms of Fibonacci numbers.

Before proceeding, we mention some notation and terminology that we will use. We denote by F_i the ith Fibonacci number, where $F_1 = F_2 = 1$, and we denote by F the set of all Fibonacci numbers $\{1, 2, 3, 5, 8, \ldots\}$. By $\mathcal{F}$ we shall mean the family of all increasing sequences of positive integers that satisfy the Fibonacci recurrence; i.e., $\mathcal{F} = \{\{a_i\}_{i=1}^{k} : a_i = a_{i-1} + a_{i-2} \text{ for } 3 \le i \le k, \text{ and } k \in \mathbb{Z}^+\}$. If S is a set of positive integers, an *S-diffsequence* is a sequence $x_1, x_2, \ldots$ such that $x_i - x_{i-1} \in S$ for all $i \ge 2$. We will find it convenient to refer to a partition of a set as a *coloring*; specifically, for a positive integer r, an *r-coloring* of a set S is any function $\chi : S \to C$, where $|C| = r$. If χ is a coloring of a set S, and $A \subset S$, we say that A is *monochromatic* under χ if χ is constant on A. We will often denote a particular coloring, where the colors are 0 and 1, as a string of 0s and 1s. Further, if T is such a string,

The fourth and fifth authors were partially supported by NSF grant DMS-0649794.

the notation T^n will denote the string $\underbrace{TT\dots T}_{n}$. Finally, for $a < b$, we will denote the set of integers $\{a, a+1, \dots, b\}$ by $[a, b]$.

For a fixed positive integer r, if $\mathcal{A}$ is a family such that $n(\mathcal{A}, k; r)$ exists for all $k \in \mathbb{Z}^+$, we say that $\mathcal{A}$ is *r-regular*. If the family $\mathcal{A}$ is r-regular for all r, we say that $\mathcal{A}$ is *regular*. If $\mathcal{A}$ is not regular, then the *degree of regularity* of $\mathcal{A}$, which is denoted by $\text{dor}(\mathcal{A})$, is the largest r such that $\mathcal{A}$ is r-regular.

In this paper we shall look at the Ramsey properties of the following three particular families of sequences:

(1) The family $\mathcal{F}$.
(2) The family AP_F consisting of all arithmetic progressions $x, x+d, x+2d, \dots$ with the property that $d \in F$.
(3) The family of F-diffsequences.

2. The Family of Fibonacci Sequences

For Fibonacci sequences of length three, the existence of the Ramsey number $n(\mathcal{F}, 3; r)$ is immediate from Schur's Theorem, since we see that this number has the same meaning as the Schur number $s(r)$. On the other hand, the following result shows that $n(\mathcal{F}, k; r)$ does not exist if $k \geq 4$, even when $r = 2$ colors.

Theorem 2.1. *There exists a 2-coloring of $\mathbb{Z}^+$ that avoids 4-term monochromatic members of $\mathcal{F}$.*

Proof. For each positive integer i, let $B_i = [2^{i-1}, 2^i - 1]$. Let χ be the 2-coloring of $\mathbb{Z}^+$ defined by $\chi(B_i) = 0$ if i is odd, and $\chi(B_i) = 1$ if i is even.

For a contradiction, assume that $\{a_1, a_2, a_3, a_4\}$ is a monochromatic Fibonacci sequence. Note that if $a_1, a_2 \in B_j$ for some j, then a_3 would belong to B_{j+1}; this is not possible since the a_i's have the same color. Hence, a_1 and a_2 belong to different blocks. Likewise, a_2 and a_3 cannot belong to the same block.

Since $\chi(a_1) = \chi(a_2)$, we must have $a_1 \in B_j$ and $a_2 \in B_k$ for some j and k such that $k \geq j + 2$. Hence, $a_1 + a_2 = a_3$ must belong to B_k since it has the same color as a_2. This contradicts our earlier statement that a_2 and a_3 are not in the same block. $\square$

Remark. If we use a second order linear homogeneous recurrence different from the Fibonacci recurrence, in many instances the associated Ramsey-type function does exist. For example, since all arithmetic progressions are solutions to the recurrence $a_n = 2a_{n-1} - a_{n-2}$, by van der Waerden's Theorem, every finite coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic solutions to this recurrence. Results on Ramsey properties for the family of Fibonacci sequences, but where the first two terms of the sequence are arbitrary positive integers (not necessarily non-decreasing) may be found in [3]. Other positive Ramsey results concerning sequences defined by second order recurrences are given in [4].

3. Arithmetic Progressions Whose Gaps are Fibonacci Numbers

In [2] the following theorem is proven:

Theorem 3.1. *If $S = \{a_i\}_{i=1}^{\infty}$ is any set of positive integers with the property that, for some real number $c > 1$, $a_i \geq ca_{i-1}$ for all but a finite number of i, then AP_S is not regular.*

As an immediate corollary, we know that AP_F is not regular, since the Fibonacci numbers have the asymptotic ratio $a_i/a_{i-1} \sim (1+\sqrt{5})/2$. By directly applying the method of proof of Theorem 3.1, as presented in [2], specifically to the sequence F, one finds that $\mathrm{dor}(AP_F) \leq 7$. By employing a different line of reasoning, we obtain Proposition 3.3 below, which gives a stronger result. The proof makes use of the following three facts from [2].

Lemma 3.2. *([2])*

(i) *If AP_S is 2-regular, then S contains a multiple of every positive integer.*

(ii) *If $B = \{a_i\}_{i=1}^{\infty}$ is a sequence of positive integers such that $a_i \geq 3a_{i-1}$ for all $i \geq 2$, then AP_B is not 2-regular.*

(iii) *If X and Y are sets of positive integers such that AP_X is not r-regular and AP_Y is not s-regular, then $AP_{X\cup Y}$ is not rs-regular.*

Proposition 3.3. *$\mathrm{dor}(AP_F) \leq 3$.*

Proof. Partition the Fibonacci sequence into the sets $A = \{x \in F : x \text{ is odd}\}$ and $B = \{x \in F : x \text{ is even}\}$. By Lemma 3.2 (i), AP_A is not 2-regular. Also, $B = \{F_{3i} : i \geq 1\}$, and since $F_k \geq \frac{3}{2}F_{k-1}$ for all $k \geq 3$, we see that B satisfies the hypothesis of Lemma 3.2 (ii). Hence, AP_B is not 2-regular. Therefore, by Lemma 3.2 (iii), AP_F is not 4-regular. □

Remark. It is shown in [2] that if m is any fixed nonnegative integer, and if $G = \cup_{n=1}^{\infty}[F_n, F_n + m]$, then AP_G is not regular.

4. The F-Diffsequences

When discussing the degree of regularity of families of diffsequences, we adopt the terminology of "accessibility" from [5]. That is, if D is a set, we shall refer to the degree of regularity of the family of D-diffsequences as the *degree of accessibility* of D, denoted by $\mathrm{doa}(D)$. If the family of D-diffsequences is regular, we will simply say that D is *accessible*.

It is obvious from the definitions that for all D, $\mathrm{doa}(D) \geq \mathrm{dor}(AP_D)$. In [5] it is remarked that F is 2-accessible (this can easily be proved by induction on the length of the diffsequence), so that $\mathrm{doa}(F) \geq 2$; but an upper bound on $\mathrm{doa}(F)$ was not known. In the following theorem, we prove that $\mathrm{doa}(F) \leq 5$. The proof makes use of the following three lemmas. Lemma 4.1 is well-known and easily proved, and Lemma 4.2 is due to T.C. Brown.

Let $\alpha = \frac{\sqrt{5}-1}{2}$, and denote by g the function defined on $\mathbb{Z}^+$ by $g(m) = 4m + 2\lfloor m\alpha \rfloor$.

Lemma 4.1. *For any $n \geq 0$, $2 \cdot \sum_{i=0}^{n} F_{3i+1} = F_{3n+3}$.*

The following lemma is an immediate consequence of a result due to T.C. Brown[1] [1].

Lemma 4.2. *([1]) For any $N \geq 2$, if $N - 1 = F_{i_1} + F_{i_2} + \cdots + F_{i_k}$ for some $i_1, i_2, \ldots, i_k$, where $i_{j+1} \geq i_j + 2$ for all $1 \leq j \leq k-1$ and $i_1 \geq 2$, then*

$$\lfloor N\alpha \rfloor = F_{i_1-1} + F_{i_2-1} + \cdots + F_{i_k-1}.$$

Lemma 4.3. *For any $n \in \mathbb{Z}^+$,*

$$g\left(\sum_{i=1}^{n} F_{3i-1}\right) = F_{3n+3} - 4 \text{ and } g\left(1 + \sum_{i=1}^{n} F_{3i-1}\right) = F_{3n+3} + 2.$$

[1] Lemma 4.2 follows by taking $\alpha = \frac{\sqrt{5}-1}{2}$ in Theorem 3 of [1].

Proof. Since $g(1) = 4$ and $g(2) = 10$, the claim is true for $n = 1$. Now, by Lemma 4.2, for $n \geq 2$,

$$\begin{aligned} g\left(\sum_{i=1}^{n} F_{3i-1}\right) &= 4 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \left\lfloor \alpha \cdot \sum_{i=1}^{n} F_{3i-1} \right\rfloor \\ &= 4 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \sum_{i=2}^{n} F_{3i-2}. \end{aligned} \tag{4.1}$$

This, along with Lemma 4.1 and the fact that for all positive integers m, $F_{3m-2} + 2F_{3m-1} = F_{3m+1}$, implies that

$$\begin{aligned} g\left(\sum_{i=1}^{n} F_{3i-1}\right) &= 4 + 2 \cdot \sum_{i=2}^{n} F_{3i+1} = 4 + F_{3n+3} - 2 \cdot (F_1 + F_4) \\ &= F_{3n+3} - 4. \end{aligned}$$

Also, for $n \geq 2$, using (4.1) we have

$$\begin{aligned} g\left(1 + \sum_{i=1}^{n} F_{3i-1}\right) &= 4 + 4 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \left\lfloor \alpha \cdot \left(1 + \sum_{i=1}^{n} F_{3i-1}\right) \right\rfloor \\ &= 4 + 4 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \sum_{i=1}^{n} F_{3i-2} \\ &= g\left(\sum_{i=1}^{n} F_{3i-1}\right) + 6 = F_{3n+3} + 2, \end{aligned}$$

which completes the proof. □

We now have the tools needed to prove the following theorem.

Theorem 4.4. *The degree of accessibility of F is at most five.*

Proof. To prove the theorem, we give a 6-coloring of $\mathbb{Z}^+$ that avoids 2-term monochromatic F-diffsequences.

Note first that for any $m \geq 1$,

$$\{g(m), g(m) + 2\} \cap F = \emptyset. \tag{4.2}$$

If (4.2) were false, then since $g(m) \geq 4$ is even, there is an $n \geq 2$ such that $g(m) = F_{3n}$ or $g(m) + 2 = F_{3n}$. From Lemma 4.3 and the fact that g is an increasing function it would then follow that

$$\sum_{i=1}^{n} F_{3i-1} < m < 1 + \sum_{i=1}^{n} F_{3i-1},$$

which is impossible.

Let the sequence a_n be defined by

$$a_n = 4 + 2\left(\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor\right).$$

Clearly, $a_n \in \{4, 6\}$ for all $n \in \mathbb{Z}^+$. For $n \in \mathbb{Z}^+$ we define $t_n = 1 + \sum_{i=1}^{n-1} a_i$.

Let C be the 6-coloring of $\mathbb{Z}^+$ determined by the partition consisting of the sets $C_1, \ldots, C_6$ defined as follows. Let $C_1 = \{t_n : n \in \mathbb{Z}^+\}$ and, for $2 \le i \le 6$, let $C_i = (i-1+C_1) - \bigcup_{j=1}^{i-1} C_j$. Clearly,

$$k \neq l \Rightarrow C_k \cap C_l = \emptyset.$$

Since gaps between any two consecutive elements of C_1 are 4 or 6, for any positive integer n there are $k \in \mathbb{Z}^+$ and $i \in [1,6]$ so that $n = t_k + i - 1$. Thus, $\mathbb{Z}^+ = \cup_{i=1}^6 C_i$.

Next, note that to prove the theorem, it is enough to show that there is no 2-term F-diffsequence contained in C_1. Moreover, since all elements of C_1 are odd, it is enough to prove that for any positive integers m and n, $n < m$,

$$t_m - t_n \notin \{F_{3i} : \ i \in \mathbb{Z}^+\}.$$

Let $n < m$, and let $N = m - n$. Since for any two nonnegative real numbers x and y, $\lfloor x + y \rfloor - (\lfloor x \rfloor + \lfloor y \rfloor) \in \{0, 1\}$, we have

$$\begin{aligned} t_m - t_n &= t_{n+N} - t_n = \sum_{i=n}^{n+N-1} a_i \\ &= 4N + 2\left(\lfloor (n+N)\alpha \rfloor - \lfloor n\alpha \rfloor\right) \in \{g(N), g(N)+2\}. \end{aligned}$$

Hence, by (4.2), $t_m - t_n \notin F$. Therefore, C_1 does not contain any 2-term F-diffsequence, which completes the proof. □

Denote by $\Delta(F, k; r)$ the least positive integer Δ (if it exists) such that every r-coloring of $[1, \Delta]$ admits a monochromatic k-term F-diffsequence. It is noted in [5] that $\Delta(F, k; 2) \le F_{k+3} - 2$. Based on the known values of Δ (see Table 1 below), this (exponential) upper bound seems very weak. The following result gives a linear lower bound on $\Delta(F, k; 2)$ for $k \ge 8$, which coincides precisely with all known values of $\Delta(F, k; 2)$ for $k \ge 8$.

Theorem 4.5. *Let $k \ge 8$. Then*

$$\Delta(F, k; 2) \ge \begin{cases} \frac{16}{3}k - 20 & \text{if } k \equiv 0 (\text{mod } 3) \\ \frac{16}{3}k - \frac{64}{3} & \text{if } k \equiv 1 (\text{mod } 3) \\ \frac{16}{3}k - \frac{65}{3} & \text{if } k \equiv 2 (\text{mod } 3) \end{cases}.$$

Proof. Case 1. $k \equiv 0 (\text{mod } 3)$. Let $n = k/3 - 2$. We give a 2-coloring of $[1, 16n+11]$ that avoids monochromatic k-term F-diffsequences, which implies that $\Delta(F, k; 2) \ge 16n + 12 = \frac{16}{3}k - 20$.

Let X denote the following 2-coloring of [1,16]: 0111010010001011. Now define

$$A_n = X^n \ 01110100100,$$

a 2-coloring of $[1, 16n + 11]$. Let $\{a_i\}_{i=1}^{8n+6}$ be the elements of $[1, 16n + 11]$ having color 0, listed in increasing order. Let $g_i = a_{i+1} - a_i$. Note that

$$\{g_i\}_{i=1}^{8n+5} = 4(21211234)^n 2121. \tag{4.3}$$

Let $x_1 < x_2 < \cdots < x_t$ be an F-diffsequence of color 0. Then for each j, $1 \le j \le t-1$, $x_{j+1} - x_j$ is a sum of consecutive g_i's, say $x_{j+1} - x_j = g_{j_1} + g_{j_1+1} + \cdots + g_{j_2}$, where $(j+1)_1 = j_2 + 1$ for all j, $1 \le j \le t-1$.

We consider two subcases. First, assume $x_1 = 1$. In this case, we see that $x_2 - x_1 \ge 13$ and $j_2 - j_1 \ge 6$. Likewise, from (4.3) we have that for each g_i, $1 < i < 8n+1$, that equals 4, at least five other summands must be added to g_i to produce a Fibonacci number. Finally, note that the last six g_i's can be summands of no $x_{j+1} - x_j$ other than possibly $x_t - x_{t-1}$. Therefore, since the sequence $\{g_i\}$ has $8n+5$ terms, it follows that

$$t - 1 \leq (8n+5) - 5(n+1) = 3n+3.$$

Now assume $x_1 \neq 1$. Then g_1 does not appear as a summand in any $x_{j+1} - x_j$. Also (as in the argument used for the previous subcase) for each $g_i = 4$ where $i > 1$, if g_i is a summand for some $x_{j+1} - x_j$, then there must be at least five other summands along with it. Hence, in this case

$$t - 1 \leq (8n+5) - 1 - 5n = 3n+4.$$

In both subcases, we have that no F-diffsequence in color 0 has length greater than $3n+5 = k-1$.

Now consider the elements, $\{b_i\}$, of color 1 under the coloring A_n. The sequence of gaps, $\{b_{i+1} - b_i\}$, is $(11234212)^n$ 1123. As was the case with color 0, each gap that equals 4 must be added to at least five adjacent gaps to yield a Fibonacci number. So the largest t such that $x_{i+1} - x_i \in F$ for all i, $1 \leq i \leq t-1$, satisfies $t - 1 \leq 8n + 4 - 5n = 3n+4$, as desired.

Case 2. $k \equiv 1 (\bmod 3)$. Let $n = (k-7)/3$. Consider the coloring $B_n = 01A_n01$ of $[1, 16n+15]$. Let $x_1 < x_2 < \cdots < x_t$ be an F-diffsequence that is monochromatic of color 0. The sequence of gaps, $\{g_i\}$, between the consecutive elements of color 0 is $24(21211234)^n 21211$. If $x_1 = 1$, then we see, using the earlier argument concerning the gaps that equal 4, there are at most $8n + 7 - 5(n+1) = 3n+2$ terms $x_{i+1} - x_i$. If $x_1 \neq 1$, then by the result for A_n in color 0, there are most $(3n+4)+1$ terms $x_{i+1} - x_i$. In both cases, no F-diffsequence of color 0 has more than $3n+6 = k-1$ terms.

Now consider the elements having color 1 under B_n. By the proof of Case 1 for the color 1, it is clear that, under B_n, there are at most $(3n+5)+1$ elements in any F-diffsequence of color 1.

Hence, we have $\Delta(F, k; 2) \leq 16n + 16 = (16/3)k - 64/3$.

Case 3: $k \equiv 2 (\bmod\ 3)$. Let $n = (k-5)/3$. Define $C_n = B_{n-1}01110$, a coloring of $[1, 16n+4]$. The sequence of gaps between the consecutive elements of color 0 is the same as that for B_{n-1} with 2,4 appended to the end of the sequence. Hence, the longest possible sequence of gaps between terms of any F-diffsequence of color 0 is $3(n-1)+5+1 = 3n+3 = k-2$. For color 1, the gap sequence is the same as that for B_{n-1} with 2,1,1 appended. Thus, this gap sequence ends with the sequence 4,2,1,1. This implies, using the proof of Case 2, color 1, that the longest possible sequence of Fibonacci gaps is $3(n-1)+5+1$ (by taking the last gap to be 8). As with color 0, the longest possible F-diffsequence of color 1 is $k-1$. Hence, $\Delta(F, k; 2) \leq 16n + 5 = (16/3)k - 65/3$. □

We conclude this section with a table of computer-generated values of $\Delta(F, k; r)$.

$r \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13
2	3	5	9	11	15	19	21	28	32	37	44	48
3	4	13	27	59								
4	9	>600										
5	13											

Table 1. Values of $\Delta(F, k; r)$

5. Final Remarks and Questions

From Section 3, we know that $1 \leq \operatorname{dor}(AP_F) \leq 3$, It has been conjectured [2] that when S is a set of positive integers such that AP_S is 2-regular, then AP_S must be regular. If this conjecture is true then $\operatorname{dor}(AP_F) = 1$.

In Section 4, we showed that $2 \leq \operatorname{doa}(F) \leq 5$. From Table 1, we see that there is a huge jump between $\Delta(F, 2; 4)$ and $\Delta(F, 3; 4)$, and we suspect that $\Delta(F, 3; 4)$ does not exist which would imply that $\operatorname{doa}(F) \leq 3$.

There is a great discrepancy between the established upper and lower bounds for $\Delta(F, k; 2)$. We wonder if the lower bounds of Theorem 4.5 represent the exact values of this function.

The proof that $\Delta(F, k; 2) \leq F_{k+3} - 2$, which consists of an obvious induction, can be applied to any Fibonacci sequence, i.e., any sequence $A = \{a_i\}$ satisfying $a_i = a_{i-1} + a_{i-2}$, for example the Lucas numbers. It would be interesting to study a table of exact values of $\Delta(A, k; 2)$ for some choices of $A \neq F$.

Acknowledgements

The authors thank Derek Anderson, Megan Craven, and Aaron Robertson for their help with the computing. We also thank Tom Brown for pointing us to Lemma 4.2.

References

[1] T. C. Brown, *Descriptions of the Characteristic Sequence of an Irrational*, Canad. Math. Bull., 36 (1993), 15–21.

[2] T. C. Brown, R. L. Graham, and B. Landman, *On the Set of Common Differences in van der Waerden's Theorem on Arithmetic Progressions*, Canad. Math. Bull., 42.1 (1999), 25–36.

[3] H. Harborth and S. Maasberg, *Rado Numbers for Homogeneous Second Order Linear Recurrences–Degree of Partition Regularity*, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), Congr. Numer., 108 (1995), 109–118.

[4] B. Landman, *Ramsey Numbers Associated with Second Order Recurrences*, J. Comb. Math. Comb. Computing, 15 (1994), 119–127.

[5] B. Landman and A. Robertson, *Avoiding Monochromatic Sequences with Special Gaps*, SIAM J. Discrete Math., 21.3 (2007), 794–801.

[6] I. Schur, *Uber die Kongruenz $x^m + y^m = z^m (mod\ p)$*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 25 (1916), 114–117.

[7] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Archief voor Wiskunde, 15 (1927), 212–216.

MSC2000: 05D10, 11B39

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