ON ALMOST SUPERPERFECT NUMBERS*

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Abstract. A positive integer n is called an *almost superperfect number* if n satisfies $\sigma(\sigma(n)) = 2n - 1$, where $\sigma(n)$ denotes the sum of positive divisors of n. In this paper, we prove the following results: (1) there does not exist any even almost superperfect number; (2) if n is an almost superperfect number, then n has at least two prime factors; (3) if n is an almost superperfect number, then $\sigma(n)$ is a perfect square; (4) if n is an almost superperfect number and n is a multiple of 3, then n is a perfect square.

1. Introduction

Inspired by the failure to disprove the existence of odd perfect numbers, numerous authors have defined a number of closely related concepts, many of which seem no more tractable than the original. For example, in [1, B9], we call n an *almost superperfect number* if n satisfies $\sigma(\sigma(n)) = 2n - 1$, where $\sigma(n)$ denotes the sum of positive divisors of n. A natural problem is: do there exist almost superperfect numbers? The problem has appeared in the first edition of Guy's book [1] since 1981. But there has not been any progress on this problem. In this paper we try to deal with this problem.

In this paper, the following results are proved.

Theorem 1. *If n is an almost superperfect number, then $\sigma(n)$ is a perfect square.*

Corollary. *If n is an almost superperfect number, then n has at least two prime factors.*

Theorem 2. *There does not exist any even almost superperfect number.*

Theorem 3. *If n is an almost superperfect number and n is a multiple of 3, then n is a perfect square.*

2. Proof of the theorems

Before the proof of the main theorem, we introduce a lemma which gives an important property of almost superperfect numbers. In this paper we always use p_i, q_j to denote primes.

Lemma. *Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ ($p_1 < p_2 < \cdots < p_t$, $\alpha_i > 0$, $i = 1, 2, \ldots, t$) is an almost superperfect number, and*

$$1 + p_i + \cdots + p_i^{\alpha_i} = {q_1}^{\beta_{i1}} \cdots {q_s}^{\beta_{is}} \quad (1 \le i \le t), \tag{1}$$

where $\beta_{1j} + \cdots + \beta_{tj} > 0$ ($1 \le j \le s$), $2 \le q_1 < \cdots < q_s$.

Then

$$\prod_{i=1}^{t} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i}}\right) \prod_{j=1}^{s} \left(1 + \frac{1}{q_j} + \cdots + \frac{1}{q_j^{\beta_{1j}+\cdots+\beta_{tj}}}\right) < 2. \tag{2}$$

*Supported by the National Natural Science Foundation of China, Grant No.10771103, and the Outstanding Graduate Dissertation program of Nanjing Normal University, No. 181200000213.

Proof. By (1) and

$$\sigma(n) = \left(1 + p_1 + \cdots + p_1^{\alpha_1}\right) \cdots \left(1 + p_t + \cdots + p_t^{\alpha_t}\right),$$

we have

$$\sigma(n) = {q_1}^{\beta_{11}+\cdots+\beta_{t1}} \cdots {q_s}^{\beta_{1s}+\cdots+\beta_{ts}}.$$

Since n is an almost superperfect number, by $\sigma(\sigma(n)) = 2n - 1$, we have

$$\frac{\sigma(\sigma(n))}{\sigma(n)} \cdot \frac{\sigma(n)}{n} = \frac{\sigma(\sigma(n))}{n} = 2 - \frac{1}{n}. \tag{3}$$

Noting the fact that $\sigma(n)/n$ is a multiplicative function whose value in the prime power p^α is $1 + 1/p + \cdots + 1/p^\alpha$, we know that the value of the left side of equation (3) is

$$\prod_{i=1}^{t} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i}}\right) \prod_{j=1}^{s} \left(1 + \frac{1}{q_j} + \cdots + \frac{1}{q_j^{\beta_{1j}+\cdots+\beta_{tj}}}\right).$$

By (3) the lemma is obviously proved. □

Proof of Theorem 1. Suppose that n is an almost superperfect number. Let the notations be as in the lemma. Then

$$\begin{aligned} 2n - 1 &= 2p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} - 1 \\ &= \left(1 + q_1 + \cdots + q_1^{\beta_{11}+\cdots+\beta_{t1}}\right) \cdots \left(1 + q_s + \cdots + q_s^{\beta_{1s}+\cdots+\beta_{ts}}\right). \end{aligned} \tag{4}$$

From equation (4) we have that $1 + q_i + \cdots + q_i^{\beta_{1i}+\cdots+\beta_{ti}}$ is an odd number for $1 \le i \le s$. So we get the fact that if $q_i > 2$ then $\beta_{1i} + \cdots + \beta_{ti}$ is even for $1 \le i \le s$.

Now, we want to prove that $\beta_{1i} + \cdots + \beta_{ti}$ is even when $q_i = 2$. Obviously, the only possibility is $i = 1$. Suppose that $q_1 = 2$ and $\beta_{11} + \cdots + \beta_{t1}$ is not even. Then we have $3 \mid 2^{\beta_{11}+\cdots+\beta_{t1}+1} - 1$.

By $1 + 2 + \cdots + 2^{\beta_{11}+\cdots+\beta_{t1}} = 2^{\beta_{11}+\cdots+\beta_{t1}+1} - 1$ and (4), we have

$$3 \mid 2p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} - 1.$$

Then

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \equiv -1 \pmod 3.$$

So there exists at least one $i (1 \le i \le t)$ satisfying $p_i^{\alpha_i} \equiv -1 \pmod 3$. Namely, $p_i \equiv -1 \pmod 3$ and α_i is an odd number.

Hence,

$$1 + p_i + \cdots + p_i^{\alpha_i} \equiv 0 \pmod 3.$$

So by (1) we have $q_2 = 3$. By (2), we have

$$\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{\beta_{11}+\cdots+\beta_{t1}}}\right)\left(1 + \frac{1}{3}\right) < 2.$$

This is impossible. So $\beta_{11} + \cdots + \beta_{t1}$ is even.

Thus we have proved that $2 \mid \beta_{1i} + \cdots + \beta_{ti}$ for $1 \le i \le s$. Since

$$\sigma(n) = {q_1}^{\beta_{11}+\cdots+\beta_{t1}} \cdots {q_s}^{\beta_{1s}+\cdots+\beta_{ts}},$$

$\sigma(n)$ is a perfect square. This completes the proof of Theorem 1. □

Proof of the Corollary. Suppose that $n = p^\alpha$(p is a prime) is an almost superperfect number.

By Theorem 1, we have $\sigma(p^\alpha) = m^2$.

Namely,

$$1 + p + \cdots + p^\alpha = m^2.$$

Ljunggren [2] proved that

$$\frac{x^n - 1}{x - 1} = y^2$$

has only two solutions $(x, y, n) = (3, 11, 5), (7, 20, 4)$. We can check that neither $p = 3, \alpha = 4$ nor $p = 7, \alpha = 3$ satisfying $\sigma(\sigma(p^\alpha)) = 2p^\alpha - 1$. This completes the proof of the corollary. □

Proof of Theorem 2. Let the notations be as in the lemma. If n is an almost superperfect number and n is an even number, then we can assume $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ $(3 \le p_2 < \cdots < p_t$, $\alpha_i > 0$, $i = 1, 2, \cdots, t)$.

By

$$2^{\alpha_1+1} - 1 = 1 + 2 + \cdots + 2^{\alpha_1} = {q_1}^{\beta_{11}} \cdots {q_s}^{\beta_{1s}},$$

there exists at least one $i(1 \le i \le s)$ satisfying $q_i \mid 2^{\alpha_1+1} - 1$. Thus,

$$q_i \le 2^{\alpha_1+1} - 1.$$

Then the left side of equation (2)

$$\ge \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{\alpha_1}}\right)\left(1 + \frac{1}{q_i}\right) \ge \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1}}\left(1 + \frac{1}{2^{\alpha_1+1} - 1}\right) = 2.$$

By the lemma, it is impossible. So there does not exist any even almost superperfect number. This completes the proof of Theorem 2. □

Proof of Theorem 3. Let the notations be as in the lemma. By Theorem 2 we need only to consider odd numbers n. If n is an almost superperfect number and n is a multiple of 3, then $p_1 = 3$. By (2) we have

$$\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{q_1}\right) < 2.$$

So $q_1 > 2$. Then $\sigma(n)$ is an odd number.

By (1) and $q_1 > 2$ we have

$$1 + p_i + \cdots + p_i^{\alpha_i} \equiv 1 \pmod 2.$$

Since $p_i > 2(1 \le i \le t)$, we have

$$\alpha_i \equiv 0 \pmod 2 \ (1 \le i \le t).$$

Hence, n is a perfect square. This completes the proof of Theorem 3. □

Acknowledgement

I sincerely thank my supervisor Professor Yong-Gao Chen for his useful discussions and encouragement during the various stages of this work. I am also grateful to the referee for giving a simplified proof of the lemma and many valuable suggestions.

References

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[2] W. Ljunggren, *Noen Setninger om Ubestemte Likninger av Formen* $(x^n - 1)/(x - 1) = y^q$, Norsk. Mat. Tidsskr. 1. Hefte, **25** (1943), 17–20.

MSC2000: 11A25,11A51,11D41

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