THE UNBOUNDEDNESS OF A FAMILY OF DIFFERENCE EQUATIONS OVER THE INTEGERS

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Abstract. In this paper, we prove that positive integer solutions $\{a_n\}$ to

$$a_n = \begin{cases} \frac{c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}}{d}, & \text{if } d|c_1a_{n-1}+\cdots+c_ka_{n-k}; \\ c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}, & \text{otherwise,} \end{cases}$$

where the c's are nonnegative integers, and $d = c_1 + c_2 + \cdots + c_k$, have the property that either $\{a_n\}$ is periodic with period at most k, or $\{a_n\}$ is unbounded.

1. Introduction

Within the field of different equations is a subfield that deals with integer solutions to recurrence relations. The Fibonacci numbers are probably the most studied example, but the Fibonacci relation is linear, which greatly simplifies the study. There is a class of nonlinear recurrence relations in which very little is known. The simplest example of something in that class is the relation defined by

$$a_n = \begin{cases} \frac{a_{n-1}+a_{n-2}}{2}, & \text{if } 2|a_{n-1}+a_{n-2}; \\ a_{n-1}+a_{n-2}, & \text{otherwise,} \end{cases} \tag{1}$$

with a_0 and a_1 positive integers. Already with this modification, things are difficult. For certain values of the initial conditions, the sequence is stationary, for example $a_0 = 1$ and $a_1 = 1$. It is proved in [1] that if the sequence is not stationary, it is unbounded. This appears to be the extend of the knowledge on this simple nonlinear variation on the Fibonacci sequence. In particular, the author is unaware of any formulas for a_n in cases where it is unbounded, or even any asymptotics on the growth of the sequence.

In this paper, we extend what is known about (1) to difference equations of the form

$$a_n = \begin{cases} \frac{c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}}{d}, & \text{if } d|c_1a_{n-1}+\cdots+c_ka_{n-k}; \\ c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}, & \text{otherwise,} \end{cases} \tag{2}$$

where the c's are nonnegative integers, and $d = c_1 + c_2 + \cdots + c_k$. Here are our main results.

Theorem 1. *Let $c_1, c_2, \ldots, c_k$ be positive integers, and let $d = c_1 + c_2 + \cdots + c_k$. Consider the recurrence relation*

$$a_n = \begin{cases} \frac{c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}}{d}, & \text{if } d|c_1a_{n-1}+\cdots+c_ka_{n-k}; \\ c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}, & \text{otherwise,} \end{cases}$$

subject to the initial condition that $a_0, a_1 \ldots, a_{k-1}$ be positive integers. This recurrence has the property that $\{a_n\}$ is either unbounded or stationary.

Theorem 1 is, itself, a special case of the following theorem.

Theorem 2. *Suppose that $c_1, c_2, \ldots, c_k$ are nonnegative integers, $c_k > 0$, and at least one other $c_i > 0$. Let $d = c_1 + c_2 + \cdots + c_k$. Consider the recurrence relation*

$$a_n = \begin{cases} \frac{c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}}{d}, & \text{if } d | c_1 a_{n-1} + \cdots + c_k a_{n-k}; \\ c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, & \text{otherwise,} \end{cases}$$

with initial condition that $a_0, a_1, \cdots, a_{k-1}$ be positive integers. This recurrence has the property that $\{a_n\}$ is either unbounded or periodic with period $\leq k$. In particular, if the recurrence does not have the form $a_n = f(a_{n-l}, a_{n-2l}, \ldots, a_{n-ml})$ for some $l > 1$, the recurrence is either unbounded or stationary.

We state Theorem 1 separately from Theorem 2 because Theorem 1 is substantially easier to prove than Theorem 2, and is a needed result on the way to the proof of Theorem 2.

Our proof makes use of an auxiliary sequence $\{b_n\}$ defined as follows:

$$b_n = 0, \quad \text{if} \quad 0 \leq n \leq k-2,$$

$$b_n = \min(a_n, a_{n-1}, \ldots, a_{n-k+1}), \quad \text{if} \quad n \geq k-1.$$

Note that we can replace the condition that $\{a_n\}$ is stationary by the weaker condition that $\{a_n\}$ is eventually stationary. That is, the only eventually stationary sequences of positive integers that satisfy (2) are strictly stationary. To see this suppose that $a_n = a_{n-1} = \ldots = a_{n-k+1} = a$ for some n. Since $a > 0$, each $c_i \geq 0$, and at least two c's are positive, $c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} > a$. Consequently,

$$a = a_n = \frac{c_1 a + c_2 a + \cdots + c_{k-1} a + c_k a_{n-k}}{c_1 + c_2 + \cdots + c_k}.$$

Solving this for a_{n-k}, which we can do since we assume $c_k \neq 0$, we have that $a_{n-k} = a$. Thus, we can induct all the way to $a_0 = a$.

2. Preliminaries

We make frequent use of the following inequalities, which we state for completeness.

Lemma 1. *Given any real numbers $x_1, \ldots, x_k$,*

(a) *If $c_1, c_2 \ldots, c_k$ are positive real numbers,*

$$\min(x_1, x_2, \ldots, x_k) \leq \frac{c_1 x_1 + c_2 x_2 + \cdots + c_k x_k}{c_1 + c_2 + \cdots + c_k} \leq \max(x_1, x_2, \ldots, x_k),$$

and both inequalities are strict unless $x_1 = x_2 = \cdots = x_k$.

(b) *If $c_1, c_2, \ldots, c_k$ are nonnegative real numbers with a positive sum, then*

$$\min(x_1, x_2, \ldots, x_k) \leq \frac{c_1 x_1 + c_2 x_2 + \cdots + c_k x_k}{c_1 + c_2 + \cdots + c_k}.$$

Proof. These are obvious. □

Lemma 2. *Given a sequence $\{a_n\}$ satisfying (2), the companion sequence $\{b_n\}$ is nondecreasing.*

Proof. The result being obvious for $0 \le n \le k-1$, we proceed by induction on n. We have

$$b_{n+1} = \min(a_{n+1}, a_n, a_{n-1}, \ldots, a_{n-k+2}).$$

Now,

$$a_{n+1} \ge \frac{c_1 a_n + c_2 a_{n-1} + \cdots + c_k a_{n-k+1}}{c_1 + c_2 + \cdots + c_k} \ge \min(a_n, a_{n-1}, \ldots, a_{n-k+1}) = b_n.$$

Thus, each of $a_{n+1}, a_n, \ldots, a_{n-k+2}$ is at least b_n, so $b_{n+1} \ge b_n$. □

When some of the c's can be zero, there is the possibility of non-stationary periodic solutions. For example,

$$a_n = \begin{cases} \frac{a_{n-2}+a_{n-4}}{2}, & \text{if } 2|a_{n-2}+a_{n-4}; \\ a_{n-2}+a_{n-4}, & \text{otherwise,} \end{cases} \tag{3}$$

has periodic solutions such as $a_{2k} = 1$, $a_{2k+1} = 2$. In this case, we could decouple the even and odd terms to get two recurrences:

$$a_{2n} = \begin{cases} \frac{a_{2(n-1)}+a_{2(n-2)}}{2}, & \text{if } 2|a_{2(n-1)}+a_{2(n-2)}; \\ a_{2(n-1)}+a_{2(n-2)}, & \text{otherwise,} \end{cases}$$

and

$$a_{2n+1} = \begin{cases} \frac{a_{2(n-1)+1}+a_{2(n-2)+1}}{2}, & \text{if } 2|a_{2(n-1)+1}+a_{2(n-2)+1}; \\ a_{2(n-1)+1}+a_{2(n-2)+1}, & \text{otherwise.} \end{cases}$$

Obviously, such a decoupling exists whenever the recurrence has the form $a_n = f(a_{n-l}, a_{n-2l}, \ldots, a_{n-ml})$ for some $l > 1$. We call such a recurrence **reducible**, and recurrences that don't have this form **irreducible**.

Finally, for this section, we state a theorem from the theory of numbers that is needed for the proof of Theorem 2.

Lemma 3. *If $n_1, n_2, \ldots, n_k$ are positive integers with $\gcd(n_1, n_2, \ldots, n_k) = 1$ then for every positive integer m there is a positive integer N such that for every integer $r \ge 0$ there are nonnegative integers $x_1, x_2, \ldots, x_k$, with $x_k \ge m$, and*

$$x_1 n_1 + x_2 n_2 + \cdots + x_k n_k = N + r. \tag{4}$$

That is, every integer $\ge N$ is a nonnegative integer linear combination of $n_1, n_2, \ldots, n_k$.

Proof. Since $\gcd(n_1, n_2, \ldots, n_k) = 1$, there are integers $y_1, y_2, \ldots, y_n$ such that $y_1 n_1 + \cdots + y_k n_k = 1$ [4, Theorem 2.15, pg. 114]. Let $y = \max(|y_1|, \ldots, |y_k|)$. We may now take $N = kyn_1 n_2 \ldots n_k + mn_k$, To see this, note that by checking the result for $r = 0, 1, 2, \ldots, n_k - 1$, we have verified the result for all nonnegative r since one may add multiples of n_k to each side of (4). Now $N = 0x_1 + \cdots + 0x_{k-1} + \frac{N}{n_k} n_k$, and $\frac{N}{n_k}$ is an integer greater than m. If $1 \le r < n_k$, then

$$ry_1 n_1 + \cdots + ry_k n_k = r.$$

We add N to both sides and note that

$$N = \frac{yn_1 n_2 \cdots n_k}{n_1} n_1 + \cdots + \frac{yn_1 n_2 \cdots n_k}{n_{k-1}} n_{k-1} + \left(\frac{yn_1 n_2 \cdots n_k}{n_k} + m \right) n_k.$$

We have

$$x_1 n_1 + x_2 n_2 + \cdots + x_k n_k = N + r,$$

with $x_i = y_i + \frac{yn_1n_2\cdots n_k}{n_i}$, if $i < k$, $x_k = y_k + \frac{yn_1n_2\cdots n_k}{n_k} + m$. Thus, each x_i is nonnegative, and $x_k \geq m$, as desired. $\square$

3. A Proof of Theorem 1

In this section, we assume that each of $c_1, c_2, \ldots, c_k$ is a positive integer. The proof of the theorem follows from the proof of the following lemma.

Lemma 4. *Given a sequence $\{a_n\}$ satisfying (2) and companion sequence $\{b_n\}$, then, $a_{n+1} \geq b_n$. Moreover, $a_{m+1} = b_m$ for some m only if $\{a_n\}$ is stationary from $n = m - k + 1$ on.*

Proof. As in Lemma 2, $a_{n+1} \geq \frac{c_1a_n+c_2a_{n-1}+\cdots+c_ka_{n-k+1}}{c_1+c_2+\cdots+c_k} \geq b_n$. By Lemma 1, the second inequality is strict unless $a_n = a_{n-1} = \cdots = a_{n-k+1}$. Thus, if for some m, $a_{m+1} = b_m$, then for some a, $a_m = \cdots = a_{m-k+1} = a$. In this case,

$$c_1a_m + c_2a_{m-1} + \cdots + c_ka_{m-k+1} = (c_1 + \cdots + c_k)a,$$

so $a_{m+1} = a$. Clearly, this gives us that $a_n = a$ for all $n \geq m - k + 1$. $\square$

Proof of Theorem. Suppose that $\{a_n\}$ is not eventually stationary. We show that for all m, $b_{m+k} > b_m$. This proves that $\{b_n\}$ is unbounded. Moreover, since $a_{n+1} \geq b_n$ for all n, $\{a_n\}$ is also unbounded.

Now $b_{m+k} = \min(a_{m+k}, a_{m+k-1}, \ldots, a_{m+1})$. If $\{a_n\}$ is not eventually stationary, then by Lemma 4, $a_{m+1} > b_m$. In fact, for all positive integers j, $a_{m+j} > b_{m+j-1} \geq b_m$. That is, each of $a_{m+1}, \ldots, a_{m+k}$ is strictly larger than b_m. Hence, b_{m+k} is strictly larger than b_m, as desired. $\square$

4. A Proof of Theorem 2

Although it must still be the case that $a_{n+1} \geq b_n$ for all n, the rest of Lemma 4 need not be true if some of the c's are zero; we must be more careful in our analysis in this case. We introduce some notation. Rewrite (2) in the form

$$a_n = \begin{cases} \frac{c_1a_{n-k}+c_2a_{n-k+l_1}+\cdots+c_ma_{n-k+l_{m-l}}}{d}, & \text{if } d| \text{ numerator;} \\ c_1a_{n-k} + c_2a_{n-k+l_1} + \cdots + c_ma_{n-k+l_{m-1}}, & \text{otherwise,} \end{cases} \tag{5}$$

where $c_1, \ldots, c_m$ are positive integers, $d = c_1 + \cdots + c_m$, and $0 < l_1 < l_2 < \cdots < l_{m-1} < k$. that is we write (2) in terms of the nonzero c's.

The key to Theorem 2 is the following lemma.

Lemma 5. *Consider the array $A_j = [a_{jk}, a_{jk+1}, \ldots, a_{jk+k-1}]$. We have that b_{jk+k-1} is the minimum value of the entries in this array. Suppose the minimum is a, and this minimum occurs r times. Then a can occur at most r times in $A_{j+1} = [a_{(j+1)k}, a_{(j+1)k+1}, \ldots, a_{(j+1)k+k-1}]$, and if a does occur r times in A_{j+1}, then either the sequence is eventually stationary or (5) is reducible.*

Proof. That a can occur at most r times in A_{j+1} follows from

$$a_{(j+1)k+i} \geq \frac{c_1a_{jk+i} + c_2a_{jk+i+l_1} + \cdots + c_ma_{jk+i+l_{m-1}}}{d}, \tag{6}$$

with strict inequality if $a_{jk+i} > a$. That is, suppose that $a_{jk+u_1}, a_{jk+u_2}, \ldots, a_{jk+u_r}$ are the terms of the sequence in A_j with values of a, where $0 \leq u_1 < u_2 < \cdots < u_r < k$. Then by

(6), $a_(j+1)k+i > a$ unless i is one of the u's. Moreover, this shows that if a does occur r times in A_{j+1}, it must be at positions $a_{(j+1)k+u_1}, a_{(j+1)k+u_2}, \ldots, a_{(j+1)k+u_r}$. Also, if

$$a = a_{(j+1)k+u_i} = \frac{c_1 a_{jk+ui} + c_2 a_{jk+ui+l_1} + \cdots + c_m a_{jk+ui+l_{m-1}}}{d}$$

then it follows that $a_{jk+ui+l_1} = \cdots = a_{jk+ui+l_{m-1}} = a$.

For the remainder of this proof, suppose that a occurs r times in A_{n+1}. Based on the discussion above, it follows that for any $n \geq jk$, $a_n = a$ if and only if

$$a_{n-k} = a_{n-k+l_1} = \cdots = a_{n-k+l_{m-1}} = a. \tag{7}$$

Also, for all $i \geq j$, A_i has exactly r values equal to a, and in exactly the positions $ik + u_1, ik + u_2, \ldots, ik + u_4$. We claim that for all nonnegative integers $x_1, x_2, \ldots, x_{m-1}$, and all $i \geq j$,

$$a_{ik+u_1+x_1l_1+x_2l_2+\cdots+x_{m-1}l_{m-1}} = a.$$

This follows by induction on $x_1 + x_2 + \cdots + x_{m-1}$, the case where the sum is 1 having already been done. The induction follows from (7). If $a_{ik+u_1+x_1l_1+x_2l_2+\cdots+x_{m-1}l_{m-1}} = a$, then $a_{(i+1)k+u_1+x_1l_1+x_2l_2+\cdots+x_{m-1}l_{m-1}} = a$. But if

$$a_{(i+1)k+u_1+x_1l_1+x_2l_2+\cdots+x_{m-1}l_{m-1}} = a,$$

then

$$\begin{gathered} a_{ik+u_1+(x_1+1)l_1+x_2l_2+\cdots+x_{m-1}l_{m-1}} = a, \\ \vdots \\ a_{ik+u_1+x_1l_1+x_2l_2+\cdots+(x_{m-1}+1)l_{m-1}} = a. \end{gathered}$$

Finally, let $D = \gcd(k, l_1, \ldots, l_{m-1})$. If $D = 1$, then by Lemma 3, there is an integer n such that we may pick $i, x_1, \ldots, x_{m-1}$ so that $i \geq j$ and $ik + u_1 + x_1l_1 + \cdots + x_{m-1}l_{m-1} = x$, for any integer $x \geq N$. Consequently, $a_x = a$ for all $x \geq N$ and the sequence of a's is eventually stationary. On the other hand, if $D > 1$, then (5) is reducible. This completes the proof of the lemma. $\square$

Proof of Theorem. Suppose first that we have an irreducible difference equation (5), with solution $\{a_n\}$. We claim that the only periodic solutions are stationary solutions. This follows from Lemma 5. As in the lemma, write $A_j = [a_{jk}, a_{jk+1}, \ldots, a_{jk+k-1}]$, with minimum entry a, and suppose that a occurs r times among the entries of A_j. Since the difference equation is irreducible, if $\{a_n\}$ is not eventually stationary, the number of occurrences of a in A_{j+1} must be strictly less than r. In particular, every entry in A_{j+r} must be larger than a since the number of occurrences of a must drop by at least 1 in each iteration. Thus, in general, the minimum value of the entries of A_j is less than the minimum value of the entries in A_{j+k}, since we always have $r < k$ for nonstationary sequences. In terms of the companion sequence $\{b_n\}$, this says that for all j, $b_{(jk+k-1)+k^2} > b_{jk+k-1}$. In fact, the proof of Lemma 5 could be strengthened to show that $b_{n+k^2} > b_n$ for all n in this case. Consequently, in the case of irreducible difference equations, any nonstationary sequence has the property that $\lim_{n\to\infty} a_n = \infty$.

Next, consider the case where (5) is reducible. This implies that $D = \gcd(k, l_1, \ldots, l_{m-1}) > 1$, and (5) may be decoupled into D recurrence relations. Since $a_n = f(a_{n-D}, a_{n-2D}, \ldots, a_{n-mD})$, we have that for each i, with $0 \leq i < D$, $a_{nD+i} = f(a_{D(n-1)+i}, \ldots, a_{D(n-m)+i})$, so letting $c_n = a_{nD+i}, c_n = f(c_{n-1}, \ldots, c_{n-m})$. Moreover, each of these D recurrence relations is irreducible. (This follows from the fact that if $\gcd(n_1, n_2, \ldots, n_k) = D$, then $\gcd(\frac{n_1}{D}, \frac{n_2}{D}, \ldots, \frac{n_k}{D}) = 1$.) If all of these derived sequences are stationary, then (5) has periodic solutions with period at most D. If any of the derived sequences is not stationary, then the sequence of the c's is unbounded, and so the sequence of the a's is also unbounded. This completes the proof. $\square$

5. Comments

Based on the proof of Theorem 2, we see that while in the case of irreducible difference equations (5), $\lim_{n\to\infty} a_n = \infty$ unless the sequence is stationary, this need not be the case for reducible difference equations. For example, with the difference equation in (3), if we have initial conditions of $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, then $a_{2n} = 1$ for all n, where as a_{2n+1} is unbounded.

It would be interesting to investigate the boundedness of solutions to (2) when the initial a's do not have to all be positive. Obviously, the case where all the a's are negative is symmetric to the case covered in this paper. The author has not investigated the case where some initial a's are positive and some are negative. See also [2] and [3] for other possible generalizations of this problem.

One might ask what happens with difference equations of the form

$$a_n = \begin{cases} \frac{c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}}{d}, & \text{if } d | c_1a_{n-1} + \cdots + c_ka_{n-k}; \\ c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}, & \text{otherwise,} \end{cases} \tag{8}$$

where $d > c_1 + c_2 + \cdots + c_k$. In particular, there is an entire family of Fibonacci-like difference equations of the form

$$a_n = \begin{cases} \frac{a_{n-1}+a_{n-2}}{m}, & \text{if } m | a_{n-1} + a_{n-2}; \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases} \tag{9}$$

with $m > 2$. These difference equations are much harder to analyze. One difficulty is that solutions to (8) or (9) may be eventually periodic, without being initially periodic. For example, in

$$a_n = \begin{cases} \frac{a_{n-1}+a_{n-2}}{3}, & \text{if } 3 | a_{n-1} + a_{n-2}; \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases} \tag{10}$$

with initial conditions $a_0 = 5$, $a_1 = 1$, the sequence continues $a_2 = 2$, $a_3 = 1$, $a_4 = 1$, $a_5 = 2$, and the solution is eventually periodic with period 3. It is conjectured in [3] that positive integer solutions to (10) are either eventually periodic with period 3, or unbounded. Scant numerical evidence by the author suggests that if the difference equation (8) is irreducible, then it is either eventually periodic with "small" period or unbounded. For example, it appears that any positive integer solution to

$$a_n = \begin{cases} \frac{a_{n-1}+a_{n-2}}{5}, & \text{if } 5 | a_{n-1} + a_{n-2}; \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

is eventually periodic with period at most 6 or it is unbounded. In the entire family in (9), the author is only aware of periodic solutions in the cases $m = 3$, $m = 5$. In particular, the author has checked that there are no periodic solutions with period < 13 for any m other than 3 or 5.

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MSC2000: 39A10, 39A11

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