ESCALATOR NUMBER SEQUENCES

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Abstract. An escalator sequence is an infinite sequence of rational numbers such that each partial sum is equal to the corresponding partial product. An escalator number is one of these partial sums, where at least the first two numbers of the sequence are included. So for each escalator sequence, there is an associated sequence of escalator numbers. In this paper we consider to what degree a single escalator number determines an associated sequence of escalator numbers in which it appears.

1. Introduction

A sequence of rational numbers, $a_1, a_2, \ldots$, is an *escalator sequence* if for each $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i.$$

The first term of the sequence, a_1, is called the *base* of the sequence. Denote the partial sums (or products) of an escalator sequence, by

$$A_n = \sum_{i=1}^{n} a_i.$$

Each A_n with $n \geq 2$ is called an *escalator number.* We call the sequence of partial sums, $A_2, A_3, \ldots$ the associated *escalator number sequence.* These concepts were first introduced by Pizá [3, 4, 5].

As shown in [2], an escalator sequence is uniquely determined by its base, $a_1 = A_1$, and any rational number other than 1 is the base of an (infinite) escalator sequence.

In this paper, we consider the possibility of reversing the process. What can we deduce about an escalator sequence, given a single escalator number in it. More specifically, given an escalator number A and some $n \geq 2$, we ask:

- Is there an escalator sequence for which $A = A_n$?
- For how many different escalator sequences does $A = A_n$?
- Is there a maximal value of $m \geq 2$ such that there is an escalator sequence for which $A = A_m$?

Define an escalator number sequence to be *maximal* if it is not a proper subsequence of any other escalator number sequence. Along with answering the above questions, we prove that each escalator number is contained in a unique maximal escalator number sequence.

2. Preliminary Results

We begin with some basic results, the proofs of which are immediate. Let $a_1, a_2, \ldots$ be an escalator sequence with associated escalator number sequence $A_2, A_3, \ldots$. Then for $n \geq 1$,

$$A_{n+1} = A_n + a_{n+1} = A_n a_{n+1}, \tag{1}$$

$$a_{n+1} = \frac{A_n}{A_n - 1}, \tag{2}$$

and

$$A_{n+1} = \frac{A_n^2}{A_n - 1}. \tag{3}$$

For each $n \in \mathbb{Z}^+$, let $u_n \in \mathbb{Z}$ and $v_n \in \mathbb{Z}^+$ with $\gcd(u_n, v_n) = 1$ such that

$$A_n = \frac{u_n}{v_n}.$$

Then

$$A_{n+1} = \frac{A_n^2}{A_n - 1} = \frac{u_n^2}{v_n(u_n - v_n)}, \tag{4}$$

with $\gcd(u_n^2, v_n(u_n - v_n)) = 1$ and so

$$u_{n+1} = \pm u_n^2 \quad \text{and} \quad v_{n+1} = v_n|u_n - v_n|. \tag{5}$$

Note that given an escalator number $A = A_n$, for each $m > n$, the values of a_m and A_m are completely determined. We now investigate the degree to which these values are determined for $m \leq n$.

Let $a_1, a_2, a_3, a_4, \ldots$ be an escalator sequence. Since $A_2 = a_1 + a_2 = a_1 a_2$, it's also true that $A_2 = a_2 + a_1 = a_2 a_1$. Thus, $a_2, a_1, a_3, a_4, \ldots$ is also an escalator sequence, one with the same associated escalator number sequence.

Given any escalator number, A, by definition, $A = A_n$ for some escalator sequence. By the above, there is at least one additional escalator sequence with $A_n = A$, unless $a_1 = a_2$. Restricting to $n = 2$, we show that these are the only such escalator sequences. Specifically, in the next two theorems we show that for a fixed escalator number, there exists at least one and at most two escalator sequences with the given escalator number equal to the partial sum A_2.

Theorem 1. *Let A be an escalator number. There exists an escalator sequence $b_1, b_2, \ldots$ with partial sum $B_2 = A$. Further, if $c_1, c_2, \ldots$ is an escalator sequence with partial sum $C_2 = A$, then either*

- *$b_i = c_i$, for $i \geq 1$ (that is, the two sequences are identical); or*
- *$b_1 = c_2$, $b_2 = c_1$, and $b_i = c_i$, for $i > 2$.*

Proof. Let $A = A_{n+1}$, with $n \in \mathbb{Z}^+$, be an escalator number with associated escalator sequence $a_1, a_2, \ldots$. If we set $b_1 = A_n$, then b_1 determines an escalator sequence $b_1, b_2, \ldots$ with partial sums $B_1 = b_1 = A_n$ and, by (3),

$$B_2 = \frac{A_n^2}{A_n - 1} = A_{n+1} = A.$$

Now if $c_1, c_2, \ldots$ is an escalator sequence with partial sum $C_2 = A = B_2$, then using (3),

$$\frac{c_1^2}{c_1 - 1} = A = \frac{b_1^2}{b_1 - 1},$$

and so, solving the quadratic equation for c_1, $c_1 = b_1$ or $c_1 = \frac{b_1}{b_1 - 1} = b_2$. Equality for $i > 2$ is immediate from (2) and (3). $\square$

Theorem 2. *There exists exactly one escalator sequence with partial sum* $A_2 = 0$*, exactly one escalator sequence with partial sum* $A_2 = 4$*, and for each escalator number* $A \notin \{0, 4\}$*, there are exactly two escalator sequences with partial sum* $A_2 = A$*.*

Proof. Let A be an escalator number. By Theorem 1, there is an escalator sequence, $b_1, b_2, \ldots$, with partial sum $B_2 = A$. As noted above, swapping the first two numbers in the sequence results in another escalator sequence. More precisely, letting $b'_1 = b_2$, $b'_2 = b_1$, and for $i > 2$, $b'_i = b_i$, $b'_1, b'_2, \ldots$ is an escalator sequence with partial sum $B'_2 = b'_1 + b'_2 = A$. These two escalator sequences with partial sums $B_2 = B'_2 = A$ are distinct unless $b_1 = b_2$. But, using (2), it is easily shown [2, 5] that this occurs only when $b_1 \in \{0, 2\}$ and so $A = A_{n+1} \in \{0, 4\}$. □

We have thus shown that given an escalator number A, there is a unique (unordered) pair of rational numbers, say $\{r, s\}$, such that $A = r + s = rs$. Determining whether there is an escalator sequence with partial sum $A_3 = A$ is therefore reduced to the question of whether at least one of the uniquely determined pair $\{r, s\}$ is an escalator number. Theorem 4 states that for a nonzero escalator number A, the numbers r and s cannot both be escalator numbers. The proof uses the following theorem from [2] which we reprove here for completeness.

Theorem 3. *A nonzero rational number* A *is an escalator number if and only if* $A^2 - 4A$ *is a square in* $\mathbf{Q}$*.*

Proof. If A is an escalator number, then solving $A = \frac{A_n^2}{A_n - 1}$ for A_n, we get $A_n = \frac{A \pm m}{2}$ with $m^2 = A^2 - 4A$. Conversely, if $A^2 - 4A = m^2$, let $a_1 = \frac{A+m}{2}$. Then by equation (3), $A_2 = \frac{a_1^2}{a_1 - 1} = A$. □

Theorem 4. *For a nonzero escalator number* $A_2 = a_1 + a_2 = a_1 a_2$*, the numbers* a_1 *and* a_2 *are not both escalator numbers.*

Proof. Suppose a_1 is an escalator number. Then $a_1^2 - 4a_1 = t^2$ for some $t \in \mathbf{Q}$. Now, $a_2^2 - 4a_2 = \left(\frac{a_1}{a_1 - 1}\right)^2 - \frac{4a_1}{a_1 - 1} = \frac{-3a_1^2 + 4a_1}{(a_1 - 1)^2} = \frac{-4a_1^2 - t^2}{(a_1 - 1)^2} < 0$. Thus by Theorem 3, a_2 is not an escalator number. □

Corollary 5. *Let* $A = A_n$ *be an escalator number with associated escalator sequence* $a_1, a_2, a_3, a_4, \ldots$*. If* $a_1 = a_2$*, then this is the unique escalator sequence with partial sum* $A_n = A$ *and if* $a_1 \neq a_2$*, then* $a_2, a_1, a_3, a_4, \ldots$ *is the only other escalator sequence with partial sum* $A_n = A$*.*

Since each of the escalator sequences in the above corollary yields the same escalator number sequence, the next result is immediate.

Corollary 6. *If* $A = A_n$ *is an escalator number, then there exists a unique escalator number sequence,* $A_2, A_3, \ldots$ *with* $A = A_n$*.*

3. Main Results

In this section, we prove Theorem 7 which states that for each nonzero escalator number A there is a unique maximal n for which there exists an escalator sequence $a_1, a_2, \ldots$, with partial sum $A_n = A$. We call this value the *width* of the escalator number A. Hence Theorem 7 asserts that each nonzero escalator number has finite width.

Theorem 7. *If A is a nonzero escalator number, then there is some minimal $W > 1$ such that for each $m > W$, there is no escalator sequence with $A = A_m$. Further, if $A = \frac{u}{v}$ where $u, v \in \mathbb{Z}$, $v > 0$, then $W \leq v + 1$.*

Proof. By Theorem 3, if A_n is any escalator number, then $A_n \leq 0$ or $A_n \geq 4$. Therefore, letting $A_n = \frac{u_n}{v_n}$ as in Section 2, if $A_n \neq 0$, then $|u_n - v_n| \geq 2$. Now, using equation (5), $v_{n+1} = v_n|u_n - v_n| > v_n$. Since $v_2 \geq 1$, it follows by induction that for all $n \geq 1$, $v_{n+1} \geq n$.

Thus, given an escalator number A with $A = \frac{u}{v}$ where $u, v \in \mathbb{Z}$, $v > 0$ and an escalator sequence $a_1, a_2, \ldots$, if $v < n$, then $A \neq A_{n+1}$. In particular, for $n > v + 1$, $A \neq A_n$, as desired. □

Observe that the width of the escalator number 4 is 2, since the escalator sequence with base $a_1 = 2$ yields $a_2 = 2$ and $A_2 = 4$, and 2 is not an escalator number. Now the proof of Theorem 7 gives the bound for the width of $A = 4 = \frac{4}{1}$ to be $v + 1 = 2$, the best possible bound in that case. This is, however, the only case in which the bound from Theorem 4 is the best possible. The following corollaries give more complete information. (It should be noted that the first terms of the escalator sequence with base 2 formed the answer to the problem proposed by E. D. Schell in the *American Mathematical Monthly* in 1945 [1] that originally inspired Pizá to study these numbers [5].)

Corollary 8. *Let $A = \frac{u}{v}$ be a nonzero escalator number with $u, v \in \mathbb{Z}$, $v > 0$, and* $\gcd(u, v) = 1$. *If $A = 4 = \frac{4}{1}$, then the width of A is $v + 1$. If $A = -\frac{1}{2}$, $\frac{9}{2}$, or $\frac{16}{3}$, then the width of A is v. Otherwise, the width of A is strictly less than v.*

Proof. Let the width of A be $W > 1$ and fix an escalator sequence with partial sum $A_W = A$. If $A_2 = 4$, then $A_3 = \frac{16}{3}$, and $A_4 = \frac{256}{39}$. Applying the same induction as in the proof of Theorem 7, for $n \geq 4$, $v_n \geq n + 35$ and so if $W \geq 4$, the width of A is strictly less than v.

Now assume that $A_2 \neq 4$. Since the only rational integer escalator numbers are 0 and 4 [2, 5] and $A_2 \neq 0$ or 4, $v_2 \geq 2$. Again using induction, we get that for $n \geq 2$, $v_n \geq n$. So the width of A is less than or equal to v.

Suppose additionally that the width of A is equal to v. If $A = A_2$, then we have $A = \frac{u}{2}$ for some odd integer u. By equation (5), $2 = v_1|u_1 - v_1|$. Thus $v_1 = 1$ or 2 and it follows that $A = -\frac{1}{2}$ or $\frac{9}{2}$. If, on the other hand, $A \neq A_2$, note that $v_3 = v_2|u_2 - v_2| \geq 2 \cdot 2 \geq 4$ and so by induction, for $n \geq 3$, $v_n \geq n + 1$, which is impossible since the width of A is equal to v.

Thus, in all other cases, the width of A is strictly less than v, as desired. □

Using equation (5) as in the proof of Theorem 7, but instead considering the numerator, we get the next corollary.

Corollary 9. *Let $A = \frac{r}{s}$ be an escalator number with $r, s \in \mathbb{Z}$, $s > 0$, and* $\gcd(r, s) = 1$. *Then either $r = \pm 1$ or there is some maximal $i \in \mathbb{Z}^+$ such that $r = \pm x^{2^i}$ with $x \in \mathbb{Z}$. The width of A is less than or equal to $i + 1$.*

Finally, we consider maximal escalator sequences. For a fixed escalator number A, let S_A be the set of all escalator number sequences, $A_2, A_3, \ldots$ with $A = A_i$ for some $i \geq 2$. The usual concept of *subsequence* defines a partial order on S_A. We show that S_A has a unique maximal element.

Theorem 10. *Each nonzero escalator number is contained in a unique maximal escalator number sequence.*

Proof. By Theorem 7, there is a maximal $i = W$ for which $A = A_i$ in any element of S_A. By Corollary 6, there is exactly one element of S_A with $A = A_W$. □

References

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