AN IDENTITY INVOLVING THE LUCAS NUMBERS AND STIRLING NUMBERS

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Abstract. In this paper, we obtain an identity involving the Lucas numbers and Stirling numbers.

1. Introduction and Results

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}(n \in \mathbb{N} = \{0, 1, 2, \ldots\})$ are defined by the second-order linear recurrence sequences:

$$F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1 \tag{1.1}$$

and

$$L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1, \tag{1.2}$$

respectively. Clearly, we have

$$L_{n+1} = F_{n+2} + F_n \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.3}$$

These sequences play a very important role in the study of the theory and application of mathematics. Therefore, the various properties of F_n and L_n were investigated by many authors (see [1, 2, 4, 5, 6]). The main purpose of this paper is to prove an identity involving the Lucas numbers and Stirling numbers. That is, we shall prove the following main conclusion.

Theorem. *Let $n \geq k$ $(n, k \in \mathbb{N})$. Then*

$$\sum_{\substack{v_1, \cdots, v_k \in \mathbb{N} \\ v_1 + \cdots + v_k = n}} \frac{L_{v_1} L_{v_2} \cdots L_{v_k}}{v_1 v_2 \cdots v_k} = \frac{k!}{n!} \sum_{j=k}^{n} (-1)^{j-k} (n-j)! \binom{n}{j} \binom{j}{n-j} s(j, k), \tag{1.4}$$

where the $s(n, k)$ are the Stirling numbers of the first kind defined by (see [3])

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n, k) x^k, \tag{1.5}$$

or by the following generating function

$$(\log(1+x))^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}. \tag{1.6}$$

2. Definition and Lemma

Definition. *For a real or complex parameter x, the generalized Fibonacci numbers $F_n^{(x)}$, which are defined by*

$$\left(\frac{1}{1-t-t^2}\right)^x = \sum_{n=0}^{\infty} F_n^{(x)} t^n. \tag{2.1}$$

The numbers $F_{n-1}^{(1)} = F_n$ are the ordinary Fibonacci numbers.

Lemma. *Let* $n \geq k (n \in \mathbb{N})$ *and*

$$\delta(n,k) := \sum_{j=k}^{n} (-1)^{j-k} (n-j)! \binom{n}{j} \binom{j}{n-j} s(j,k). \tag{2.2}$$

Then

$$n! F_n^{(x)} = \sum_{k=1}^{n} \delta(n,k) x^k. \tag{2.3}$$

Proof. By (2.1) and (1.5), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(x)} t^n = \left(\frac{1}{1-t-t^2} \right)^x &= \sum_{j=0}^{\infty} \binom{x+j-1}{j} (t+t^2)^j \\
= \quad & \sum_{j=0}^{\infty} \binom{x+j-1}{j} t^j \sum_{n=0}^{j} \binom{j}{n} t^n \\
= \quad & \sum_{j=0}^{\infty} \binom{x+j-1}{j} \sum_{n=j}^{2j} \binom{j}{n-j} t^n \\
= \quad & \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{j}{n-j} \binom{x+j-1}{j} t^n, \qquad (2.4)
\end{aligned}$$

which readily yields

$$\begin{aligned}
n! F_n^{(x)} &= n! \sum_{j=0}^{n} \binom{j}{n-j} \binom{x+j-1}{j} \\
= \quad & n! \sum_{j=0}^{n} \frac{1}{j!} \binom{j}{n-j} (x+j-1)(x+j-2)\cdots(x+1)x \\
= \quad & \sum_{j=0}^{n} (n-j)! \binom{n}{j} \binom{j}{n-j} \sum_{k=1}^{j} (-1)^{j-k} s(j,k) x^k \\
= \quad & \sum_{k=1}^{n} \sum_{j=k}^{n} (-1)^{j-k} (n-j)! \binom{n}{j} \binom{j}{n-j} s(j,k) x^k = \sum_{k=1}^{n} \delta(n,k) x^k.
\end{aligned}$$

This completes the proof of Lemma. □

Remark 1. *Setting* $n = 1, 2, 3, 4$ *in Lemma, we get*

$$1! F_1^{(x)} = x, 2! F_2^{(x)} = 3x + x^2, 3! F_3^{(x)} = 8x + 9x^2 + x^3,$$

and

$$4! F_4^{(x)} = 42x + 59x^2 + 18x^3 + x^4.$$

3. Proof of the Theorem

Proof of the Theorem. By applying the Lemma, we have

$$k!\delta(n,k) = n!\frac{d^k}{dx^k}F_n^{(x)}|_{x=0}. \tag{3.1}$$

On the other hand, it follows from (2.1) that

$$\sum_{n=k}^{\infty}\frac{d^k}{dx^k}F_n^{(x)}|_{x=0}t^n = \left(\log\frac{1}{1-t-t^2}\right)^k. \tag{3.2}$$

Thus, by (3.1) and (3.2), we have

$$k!\sum_{n=k}^{\infty}\delta(n,k)\frac{t^n}{n!} = \left(\log\frac{1}{1-t-t^2}\right)^k. \tag{3.3}$$

By

$$\frac{d}{dt}\log\frac{1}{1-t-t^2} = \frac{1+2t}{1-t-t^2} = \sum_{n=0}^{\infty}F_n^{(1)}t^n + 2t\sum_{n=0}^{\infty}F_n^{(1)}t^n$$

we have

$$\begin{aligned}\log\frac{1}{1-t-t^2} &= \sum_{n=0}^{\infty}F_n^{(1)}\frac{t^{n+1}}{n+1} + 2\sum_{n=0}^{\infty}F_n^{(1)}\frac{t^{n+2}}{n+2} = \sum_{n=0}^{\infty}F_{n+1}\frac{t^{n+1}}{n+1} + 2\sum_{n=0}^{\infty}F_{n+1}\frac{t^{n+2}}{n+2}\\ &= \sum_{n=0}^{\infty}F_{n+1}\frac{t^{n+1}}{n+1} + 2\sum_{n=1}^{\infty}F_n\frac{t^{n+1}}{n+1} = \sum_{n=0}^{\infty}(F_{n+1}+2F_n)\frac{t^{n+1}}{n+1} = \sum_{n=0}^{\infty}L_{n+1}\frac{t^{n+1}}{n+1}\\ &= \sum_{n=1}^{\infty}L_n\frac{t^n}{n}\end{aligned} \tag{3.4}$$

which yields

$$k!\sum_{n=k}^{\infty}\delta(n,k)\frac{t^n}{n!} = \left(\sum_{n=1}^{\infty}\frac{L_n}{n}t^n\right)^k = \sum_{n=k}^{\infty}\left(\sum_{\substack{v_1,\cdots,v_k\in\mathbb{N}\\ v_1+\cdots+v_k=n}}\frac{L_{v_1}L_{v_2}\cdots L_{v_k}}{v_1v_2\cdots v_k}\right)t^n. \tag{3.5}$$

By (3.5), we have

$$\delta(n,k) = \frac{n!}{k!}\sum_{\substack{v_1,\cdots,v_k\in\mathbb{N}\\ v_1+\cdots+v_k=n}}\frac{L_{v_1}L_{v_2}\cdots L_{v_k}}{v_1v_2\cdots v_k}. \tag{3.6}$$

By (3.6) and (2.2), we may immediately deduce the following

$$\sum_{\substack{v_1,\cdots,v_k\in\mathbb{N}\\ v_1+\cdots+v_k=n}}\frac{L_{v_1}L_{v_2}\cdots L_{v_k}}{v_1v_2\cdots v_k} = \frac{k!}{n!}\sum_{j=k}^{n}(-1)^{j-k}(n-j)!\binom{n}{j}\binom{j}{n-j}s(j,k). \tag{3.7}$$

This completes the proof of the Theorem. □

Remark 2. *Setting $k = 1$ in (3.7) and noting that $s(j, 1) = (-1)^{j-1}(j-1)! \quad (j \in \mathbb{N})$ (see [3]), we have*

$$L_n = \sum_{j=1}^{n} \frac{n}{j} \binom{j}{n-j}. \tag{3.8}$$

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