IDENTITIES INVOLVING BERNOULLI NUMBERS RELATED TO SUMS OF POWERS OF INTEGERS

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Abstract. Pointing out the relations between integer power's sums and Bernoulli and Genocchi polynomials, several combinatorial identities are established.

1. Introduction and preliminaries

In [3], the authors establish several identities involving Bernoulli numbers by means of series expansion of trigonometric functions. We are able to recover some of these results and obtain new ones, with a surprisingly easier technique.

The classical Bernoulli and Genocchi numbers are respectively defined by exponential generating functions as follows:

$$\frac{z}{e^z-1} = \sum_{n\geq 0} B_n \frac{z^n}{n!}, \quad \text{and} \quad \frac{2z}{e^z+1} = \sum_{n\geq 0} G_n \frac{z^n}{n!}.$$

According to the identity

$$\frac{z}{e^z+1} = \frac{z}{e^z-1} - \frac{2z}{e^{2z}-1},$$

$G_n = 2(1-2^n)B_n$. We can generate their associated polynomials by multiplying the generating functions by the factor e^{zx}, therefore they read as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k}, \tag{1.1}$$

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k\, x^{n-k}. \tag{1.2}$$

The Euler numbers arise from the hyperbolic secant's expansion

$$\frac{1}{\cosh z} = \frac{2e^z}{e^{2z}+1} = \sum_{n\geq 0} E_n \frac{z^n}{n!}, \tag{1.3}$$

while the Euler polynomials are generated by

$$\frac{2e^{zx}}{e^z+1} = \sum_{n\geq 0} E_n(x) \frac{z^n}{n!}. \tag{1.4}$$

It is easy to verify that, for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$

$$G_n(x) = n\, E_{n-1}(x) \quad \text{and} \quad E_n(1/2) = \frac{E_n}{2^n}.$$

Definition 1. *Let $\{P_n\}_{n=0}^{\infty}$ be the sequence defined by the following exponential generating function*

$$\frac{z}{\sinh z} \;=\; \frac{2\,z\,e^{z}}{e^{2z}-1} \;=\; \sum_{n\geq 0} P_n \frac{z^n}{n!}. \tag{1.5}$$

This sequence is easily related to Bernoulli numbers as follows:

$$P_n \;=\; 2^n B_n + G_n \;=\; 2(1-2^{n-1})B_n, \tag{1.6}$$

which can be confirmed by extracting the coefficient of $z^n/n!$ on both sides of the functional relation

$$\frac{ze^{z/2}}{e^z-1} - \frac{z}{e^z-1} \;=\; \frac{z}{e^{z/2}+1}.$$

Denoting by $P_n(x)$ the associated polynomial

$$P_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} P_k x^{n-k}, \tag{1.7}$$

it follows that

$$P_n(x) \;=\; 2^n\, B_n\left(\frac{x}{2}+\frac{1}{2}\right), \tag{1.8}$$

by comparing the generating functions of these polynomials.

2. Polynomials and sums of integer powers

In the following proposition we point out how our cited polynomials are related to the sum of integer powers.

Proposition 2. *For $\lambda \in \mathbb{N} = \{1,2,3,\ldots\}$,*

$$\sum_{k=1}^{\lambda-1} k^{n-1} \;=\; \frac{1}{n}\left\{B_n(\lambda) - (-1)^n B_n\right\}, \tag{2.1}$$

$$\sum_{k=1}^{\lambda-1} (-1)^k k^{n-1} \;=\; \frac{(-1)^{\lambda+1}}{2n}\left\{G_n(\lambda) - (-1)^{\lambda+n} G_n\right\}. \tag{2.2}$$

For $\lambda \in \mathbb{N}_0 = \{0,1,2,\ldots\}$,

$$\sum_{k=1}^{\lambda} (2k-1)^{n-1} \;=\; \frac{2^{n-1}}{n}\left\{B_n(\tfrac{1}{2}+\lambda) - B_n - \frac{G_n}{2^n}\right\}, \tag{2.3}$$

$$\sum_{k=1}^{\lambda} (-1)^k (2k-1)^{n-1} \;=\; (-1)^{\lambda}\frac{2^{n-2}}{n}\left\{G_n(\tfrac{1}{2}+\lambda) - (-1)^{\lambda}\, n\,\frac{E_{n-1}}{2^{n-1}}\right\}. \tag{2.4}$$

Proof. The proof is equivalent for each item, so we just prove the first one. Let us consider the generating function of the Bernoulli polynomials

$$B(x|z) \;=\; \frac{z}{e^z-1}\, e^{zx}$$

and the sequence produced for $x \in \mathbb{N}$. Therefore,

$$B(1|z) \;=\; \frac{ze^z}{e^z-1} \;=\; \frac{z}{e^z-1} + z \;\xrightarrow{[z^n/n!]}\; B_n + \delta_{n,1} \;=\; (-1)^n B_n,$$

$$
\begin{aligned}
B(2|z) &= \frac{ze^{2z}}{e^z-1} = \frac{ze^z}{e^z-1} + ze^z \xrightarrow{[z^n/n!]} (-1)^n B_n + n,\\
&\vdots\\
B(\lambda|z) &= \frac{ze^{\lambda z}}{e^z-1} = \frac{ze^{(\lambda-1)z}}{e^z-1} + ze^{(\lambda-1)z} \xrightarrow{[z^n/n!]} (-1)^n B_n + n\sum_{k=1}^{\lambda-1} k^{n-1}
\end{aligned}
$$

where $\lambda \in \mathbb{N}$, so (2.1) is verified by induction. □

The last proposition leads us to establish many identities involving Bernoulli numbers. In the following the more remarkable instances are displayed. Particularly, (2.5) corresponds to the results in [3, Theorem 1].

Proposition 3. *For $\lambda \in \mathbb{N}$,*

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k}}{\lambda^{2k}} B_{2k} &= \frac{2n+1}{\lambda^{2n+1}} \sum_{k=1}^{\lambda-1} \left\{1-(-1)^{k+\lambda}\right\} k^{2n}\\
&+ \frac{\left\{1-(-1)^{\lambda}\right\}\delta_{0,n}}{2\lambda}, \qquad (2.5)\\
\sum_{k=0}^{n} \binom{2n}{2k} \frac{2-2^{2k}}{\lambda^{2k}} B_{2k} &= \frac{2n}{\lambda^{2n}} \sum_{k=1}^{\lambda-1} \left\{1-(-1)^{k+\lambda}\right\} k^{2n-1}\\
&+ \left\{1+(-1)^{\lambda}\left(1-2^{2n}\right)\right\} \frac{B_{2n}}{\lambda^{2n}}. \qquad (2.6)
\end{aligned}
$$

Proof. For $\lambda \in \mathbb{N}$, we obtain the following polynomial summation

$$B_n(\lambda) + \frac{1}{2} G_n(\lambda) = \lambda^n \sum_{k=0}^{n} \binom{n}{k} \left(2-2^k\right) \frac{B_k}{\lambda^k}$$

in view of the connection between Bernoulli and Genocchi numbers. On the other hand, (2.1)-(2.2) lead us to

$$\lambda^n \sum_{k=0}^{n} \binom{n}{k} \left(2-2^k\right) \frac{B_k}{\lambda^k} = n \sum_{k=1}^{\lambda-1} \left\{1+(-1)^{k+\lambda+1}\right\} k^{n-1} + (-1)^n B_n + \frac{(-1)^{n+\lambda} G_n}{2}.$$

Splitting up this identity according to the parity of n, it is easy to obtain (2.5)-(2.6) by taking into account that $B_{2n+1} = -\delta_{n,0}/2$ and $G_{2n+1} = \delta_{n,0}$, $n \in \mathbb{N}_0$. □

For $\lambda = 1, 2, 3$ in the previous proposition, we readily obtain the following identities.

Example 4.

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{2n+1}{2k} \left(2-2^{2k}\right) B_{2k} &= \delta_{0,n}, \qquad (2.7a)\\
\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k}}{2^{2k}} B_{2k} &= \frac{2n+1}{2^{2n}}, \qquad (2.7b)\\
\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k}}{3^{2k}} B_{2k} &= \frac{2^{2n+1}}{3^{2n+1}} (2n+1) + \frac{\delta_{0,n}}{3}, \qquad (2.7c)
\end{aligned}
$$

$$\sum_{k=0}^{n} \binom{2n}{2k} (2-2^{2k})\, B_{2k} = 2^{2n} B_{2n}, \tag{2.7d}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2-2^{2k}}{2^{2k}}\, B_{2k} = \frac{4n}{2^{2n}} + \left(2-2^{2n}\right) \frac{B_{2n}}{2^{2n}}, \tag{2.7e}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2-2^{2k}}{3^{2k}}\, B_{2k} = \frac{2^{2n}}{3^{2n}} \left(2n + B_{2n}\right). \tag{2.7f}$$

Evaluating $B_n(\lambda+1/2) + \frac{1}{2}\, G_n(\lambda+1/2)$ according to (2.3)-(2.4), we obtain the following proposition.

Proposition 5. *For* $\lambda \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2^{2k}(2-2^{2k})}{(2\lambda+1)^{2k}}\, B_{2k} = \frac{4n+2}{(2\lambda+1)^{2n+1}} \sum_{k=1}^{\lambda} \left\{1+(-1)^{k+\lambda}\right\} (2k-1)^{2n} + \frac{(-1)^{\lambda}(2n+1)}{(2\lambda+1)^{2n+1}}\, E_{2n}, \tag{2.8}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2^{2k}(2-2^{2k})}{(2\lambda+1)^{2k}}\, B_{2k} = \frac{4n}{(2\lambda+1)^{2n}} \sum_{k=1}^{\lambda} \left\{1+(-1)^{k+\lambda}\right\} (2k-1)^{2n-1} + \left(2-2^{2n}\right) \frac{B_{2n}}{(2\lambda+1)^{2n}}. \tag{2.9}$$

For $\lambda = 0, 1, 2$ in the previous proposition we readily obtain the following identities.

Example 6.

$$\sum_{k=0}^{n} \binom{2n+1}{2k} 2^{2k}(2-2^{2k})\, B_{2k} = (2n+1)\, E_{2n}, \tag{2.10a}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2^{2k}(2-2^{2k})}{3^{2k}}\, B_{2k} = \frac{8n+4}{3^{2n+1}} + \frac{2n+1}{3^{2n+1}}\, E_{2n}, \tag{2.10b}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2^{2k}(2-2^{2k})}{5^{2k}}\, B_{2k} = \frac{(8n+4)\, 3^{2n}}{5^{2n+1}} + \frac{2n+1}{5^{2n+1}}\, E_{2n}, \tag{2.10c}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} 2^{2k}(2-2^{2k})\, B_{2k} = (2-2^{2n})\, B_{2n}, \tag{2.10d}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2^{2k}(2-2^{2k})}{3^{2k}}\, B_{2k} = \frac{8n}{3^{2n}} - \frac{2-2^{2n}}{3^{2n}}\, B_{2n}, \tag{2.10e}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \frac{2^{2k}(2-2^{2k})}{5^{2k}}\, B_{2k} = \frac{8n}{5^{2n}} 3^{2n-1} - \frac{2-2^{2n}}{5^{2n}}\, B_{2n}. \tag{2.10f}$$

Evaluating $B_n(2\lambda)+\frac{1}{2}\, P_n(2\lambda)$ according to (2.1)-(2.3), we obtain the following proposition.

Proposition 7. *For* $\lambda \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k-1}}{(2\lambda)^{2k}} B_{2k} &= \frac{2n+1}{(2\lambda)^{2n+1}} \left\{ \sum_{k=1}^{2\lambda-1} k^{2n} + \sum_{k=1}^{\lambda} (2k-1)^{2n} \right\} \\
&+ \frac{2n+1+\delta_{0,n}}{4\lambda}, \qquad (2.11) \\
\sum_{k=0}^{n} \binom{2n}{2k} \frac{2-2^{2k-1}}{(2\lambda)^{2k}} B_{2k} &= \frac{2n}{(2\lambda)^{2n}} \left\{ \sum_{k=1}^{2\lambda-1} k^{2n-1} + \sum_{k=1}^{\lambda} (2k-1)^{2n-1} \right\} \\
&+ \frac{2-2^{2n-1}}{(2\lambda)^{2n}} B_{2n} + \frac{n}{2\lambda}. \qquad (2.12)
\end{aligned}$$

For $\lambda = 1, 2, 3$ in the previous proposition we readily obtain the following identities.

Example 8.

$$\begin{aligned}
\sum_{k=0}^{n} \binom{2n+1}{2k} (2-2^{2k-1}) B_{2k} &= \frac{2n+1}{2^{2n}} + \frac{1}{4} (1+2n+\delta_{n,0}), \qquad (2.13a) \\
\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k-1}}{4^{2k}} B_{2k} &= \frac{4n+2}{4^{2n+1}} \left(1+2^{2n-1}+3^{2n}\right) \\
&+ \frac{1}{8} (1+2n+\delta_{n,0}), \qquad (2.13b) \\
\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k-1}}{6^{2k}} B_{2k} &= \frac{4n+2}{6^{2n+1}} \left(1+2^{2n-1}+3^{2n}+2^{4n-1}+5^{2n}\right) \\
&+ \frac{1}{12} (1+2n+\delta_{n,0}), \qquad (2.13c) \\
\sum_{k=0}^{n} \binom{2n}{2k} (2-2^{2k-1}) B_{2k} &= \frac{n}{2} + \frac{n}{2^{2n-2}} + \left(2-2^{2n-1}\right) \frac{B_{2n}}{2^{2n}}, \qquad (2.13d) \\
\sum_{k=0}^{n} \binom{2n}{2k} \frac{2-2^{2k-1}}{4^{2k}} B_{2k} &= \frac{n}{4} + \frac{2n}{2^{4n-1}} (1+2^{2n-2}+3^{2n-1}) \\
&+ \left(2-2^{2n-1}\right) \frac{B_{2n}}{4^{2n}}, \qquad (2.13e) \\
\sum_{k=0}^{n} \binom{2n}{2k} \frac{2-2^{2k-1}}{6^{2k}} B_{2k} &= \frac{n}{6} + \frac{n}{3^{2n}} (1+2^{2n-2}+3^{2n-1}+2^{4n-1}+5^{2n-1}) \\
&+ \left(2-2^{2n-1}\right) \frac{B_{2n}}{6^{2n}}. \qquad (2.13f)
\end{aligned}$$

Evaluating $B_n(2\lambda+1)+\frac{1}{2}P_n(2\lambda+1)$ according to (2.1), we obtain the following proposition.

Proposition 9. *For* $\lambda \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} \binom{2n+1}{2k} \frac{2-2^{2k-1}}{(2\lambda+1)^{2k}} B_{2k} = \frac{2n+1}{(2\lambda+1)^{2n+1}} \left\{ \sum_{k=1}^{2\lambda} k^{2n} + 2^{2n} \sum_{k=1}^{\lambda} k^{2n} \right\}$$

$$+ \quad \frac{2n+1+2\,\delta_{0,n}}{2(2\lambda+1)}, \tag{2.14}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{2-2^{2k-1}}{(2\lambda+1)^{2k}}\,B_{2k} = \frac{2n}{(2\lambda+1)^{2n}}\left\{\sum_{k=1}^{2\lambda}k^{2n-1}+2^{2n-1}\sum_{k=1}^{\lambda}k^{2n-1}\right\}$$
$$+ \quad \frac{(1+2^{2n-1})\,B_{2n}}{(2\lambda+1)^{2n}}+\frac{n}{2\lambda+1}. \tag{2.15}$$

For $\lambda = 0, 1, 2$ in the previous proposition we readily obtain the following identities.

Example 10.

$$\sum_{k=0}^{n}\binom{2n+1}{2k}(2-2^{2k-1})\,B_{2k} = \frac{1}{2}\,(1+2n+2\delta_{n,0})\,, \tag{2.16a}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k}\frac{2-2^{2k-1}}{3^{2k}}\,B_{2k} = \frac{1+2^{2n+1}}{3^{2n+1}}+\frac{1}{6}\,(1+2n+2\delta_{n,0})\,, \tag{2.16b}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k}\frac{2-2^{2k-1}}{5^{2k}}\,B_{2k} = \frac{2n+1}{5^{2n+1}}\left(1+2^{2n+1}+3^{2n}+2^{4n+1}\right)$$
$$+ \quad \frac{1}{10}\,(1+2n+2\delta_{n,0})\,, \tag{2.16c}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}(2-2^{2k-1})\,B_{2k} = n+(1+2^{2n-1})B_{2n}, \tag{2.16d}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{2-2^{2k-1}}{3^{2k}}\,B_{2k} = \frac{n}{3}+\frac{2n(1+2^{2n})}{3^{2n}}+\frac{1+2^{2n-1}}{3^{2n}}B_{2n}, \tag{2.16e}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{2-2^{2k-1}}{5^{2k}}\,B_{2k} = \frac{n}{5}+\frac{2n}{5^{2n}}\left(1+2^{2n}+3^{2n-1}+2^{4n-1}\right)$$
$$+ \quad \frac{5}{2^{2n}}\left(1+2^{2n-1}\right)B_{2n}. \tag{2.16f}$$

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