ON THE SUM OF RECIPROCAL FIBONACCI NUMBERS

HIDEYUKI OHTSUKA AND SHIGERU NAKAMURA

Abstract. In this paper we consider infinite sums derived from the reciprocals of the Fibonacci numbers, and infinite sums derived from the reciprocals of the square of the Fibonacci numbers. Applying the floor function to the reciprocals of these sums, we obtain equalities that involve the Fibonacci numbers.

1. Introduction

A great many Fibonacci and Lucas identities have been discovered so far by many amateurs and professional mathematicians. Recently, the first author has found some new sum formulas for the reciprocals of the Fibonacci numbers, and the reciprocals of the square of the Fibonacci numbers.

Our theorems are as follows:

Theorem 1.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \textit{if } n \textit{ is even and } n \geq 2; \\ F_{n-2} - 1, & \textit{if } n \textit{ is odd and } n \geq 1. \end{cases}$$

Theorem 2.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \textit{if } n \textit{ is even and } n \geq 2; \\ F_{n-1}F_n, & \textit{if } n \textit{ is odd and } n \geq 1. \end{cases}$$

We believe that these theorems involve new ideas. In this note, we shall prove these theorems.

2. Proof of Theorem 1

To prove Theorem 1, we need two lemmas.

Lemma 1.

(1) $\displaystyle\sum_{k=n}^{\infty} \frac{F_{n-2}}{F_k} < 1$, *if* n *is even and* $n \geq 2$.

(2) $\displaystyle\sum_{k=n}^{\infty} \frac{F_{n-2}}{F_k} > 1$, *if* n *is odd and* $n \geq 1$.

Proof. For $n > 0$,

$$\begin{aligned}
&\frac{1}{F_n}-\frac{2}{F_{n+2}}-\frac{1}{F_{n+3}}=\frac{F_{n+2}-2F_n}{F_nF_{n+2}}-\frac{1}{F_{n+3}}\\
=\;&\frac{F_{n-1}}{F_nF_{n+2}}-\frac{1}{F_{n+3}}=\frac{F_{n-1}F_{n+3}-F_nF_{n+2}}{F_nF_{n+2}F_{n+3}}\\
=\;&\frac{\left(F_{n+1}^2+(-1)^n F_2^2\right)-\left(F_{n+1}^2+(-1)^{n+1} F_1^2\right)}{F_nF_{n+2}F_{n+3}}\\
=\;&\frac{2\,(-1)^n}{F_nF_{n+2}F_{n+3}}.
\end{aligned}$$

(1) If n is even and $n > 0$, then

$$\frac{1}{F_n}-\frac{2}{F_{n+2}}-\frac{1}{F_{n+3}}>0.$$

Therefore,

$$\frac{1}{F_n}>\frac{1}{F_{n+2}}+\frac{1}{F_{n+2}}+\frac{1}{F_{n+3}}.$$

Using this inequality for $n > 2$ repeatedly, we have

$$\begin{aligned}
\frac{1}{F_{n-2}} &> \frac{1}{F_n}+\frac{1}{F_n}+\frac{1}{F_{n+1}}\\
&> \frac{1}{F_n}+\frac{1}{F_{n+1}}+\left(\frac{1}{F_{n+2}}+\frac{1}{F_{n+2}}+\frac{1}{F_{n+3}}\right)\\
&> \frac{1}{F_n}+\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}+\frac{1}{F_{n+3}}+\left(\frac{1}{F_{n+4}}+\frac{1}{F_{n+4}}+\frac{1}{F_{n+5}}\right)\\
&> \cdots > \frac{1}{F_n}+\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}+\frac{1}{F_{n+3}}+\frac{1}{F_{n+4}}+\frac{1}{F_{n+5}}+\cdots\\
&= \sum_{k=n}^{\infty}\frac{1}{F_k}.
\end{aligned}$$

And we have

$$\sum_{k=n}^{\infty}\frac{F_{n-2}}{F_k}<1,$$

if n is even, and $n > 2$.

This inequality holds for $n = 2$. Thus (1) is proved and (2) is proved similarly. □

Lemma 2. *For* $n \geq 1$,

(1) $\displaystyle\sum_{k=n}^{\infty}\frac{F_{n-2}-1}{F_k}<1.$

(2) $\displaystyle\sum_{k=n}^{\infty}\frac{F_{n-2}+1}{F_k}>1.$

Proof. We denote $\alpha=\dfrac{1+\sqrt{5}}{2}$ and $\beta=\dfrac{1-\sqrt{5}}{2}$.

(1) Let $k \geq m \geq 1$.
Then,

$$\begin{aligned} \sqrt{5}(F_{k-m} - \alpha^{-m}F_k) &= \alpha^{k-m} - \beta^{k-m} - \alpha^{-m}(\alpha^k - \beta^k) \\ &= \alpha^{-m}\beta^k - \beta^{k-m} \leq \alpha^{-m} \mid \beta \mid^k + \mid \beta \mid^{k-m} \\ &< \alpha^0 \mid \beta \mid^0 + \mid \beta \mid^0 = 2 < \sqrt{5}. \end{aligned}$$

Therefore,

$$F_{k-m} - \alpha^{-m}F_k < 1.$$

Therefore,

$$\frac{F_{k-m} - 1}{F_k} < \alpha^{-m}.$$

Putting $m = k - n + 2$, we have

$$\frac{F_{n-2} - 1}{F_k} < \alpha^{n-2-k} \quad (2 \leq n \leq k+1).$$

By this inequality, we have

$$\sum_{k=n}^{\infty} \frac{F_{n-2} - 1}{F_k} < \sum_{k=n}^{\infty} \alpha^{n-2-k} = \sum_{j=2}^{\infty} \alpha^{-j} = \frac{1}{\alpha^2(1-\alpha^{-1})} = \frac{1}{\alpha^2 - \alpha} = 1.$$

Thus, (1) is proved and (2) is proved similarly. □

Proof of Theorem 1.
Case 1. n **is even, and** $n \geq 2$. If $n = 2$,

$$\sum_{k=2}^{\infty} \frac{1}{F_k} > \frac{1}{F_2} = 1.$$

Therefore,

$$0 < \left(\sum_{k=2}^{\infty} \frac{1}{F_k} \right)^{-1} < 1.$$

This implies

$$\left\lfloor \left(\sum_{k=2}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = 0 = F_{2-2}.$$

If $n \geq 4$, then by our lemmas we have

$$\frac{1}{F_{n-2} + 1} < \sum_{k=n}^{\infty} \frac{1}{F_k} < \frac{1}{F_{n-2}} \ .$$

Therefore,

$$F_{n-2} < \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} < F_{n-2} + 1.$$

Therefore,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2}.$$

Case 2. If n is odd, and $n \geq 1$. $n = 1$ and 3 cases are easily verified. If $n \geq 5$, we have

$$\frac{1}{F_{n-2}} < \sum_{k=n}^{\infty} \frac{1}{F_k} < \frac{1}{F_{n-2} - 1}.$$

Therefore,

$$F_{n-2} - 1 < \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} < F_{n-2}.$$

Thus,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2} - 1.$$

This completes the proof. □

3. Proof of Theorem 2

As for Theorem 1, we require two lemmas for the proof of Theorem 2.

Lemma 3.

(1) $\displaystyle\sum_{k=n}^{\infty} \frac{F_{n-1}F_n}{F_k^2} > 1$, *if n is even and $n \geq 2$.*

(2) $\displaystyle\sum_{k=n}^{\infty} \frac{F_{n-1}F_n}{F_k^2} < 1$, *if n is odd and $n \geq 1$.*

Proof. For $n > 1$, with the use of de Moivre-Binet form, we have

$$\begin{aligned}
&\frac{1}{F_{n-1}F_n} - \frac{1}{F_n^2} - \frac{1}{F_{n+1}^2} - \frac{1}{F_{n+1}F_{n+2}} \\
= &\frac{F_n - F_{n-1}}{F_{n-1}F_n^2} - \frac{F_{n+2} + F_{n+1}}{F_{n+1}^2 F_{n+2}} \\
= &\frac{F_{n-2}}{F_{n-1}F_n^2} - \frac{F_{n+3}}{F_{n+1}^2 F_{n+2}} \\
= &\frac{F_{n-2}F_{n+1}^2 F_{n+2} - F_{n-1}F_n^2 F_{n+3}}{F_{n-1}F_n^2 F_{n+1}^2 F_{n+2}} \\
= &\frac{F_{n+1}^2(F_n^2 + (-1)^{n-1}F_2^2) - F_n^2(F_{n+1}^2 + (-1)^n F_2^2)}{F_{n-1}F_n^2 F_{n+1}^2 F_{n+2}} \\
= &\frac{(-1)^{n-1}F_{n+1}^2 - (-1)^n F_n^2}{F_{n-1}F_n^2 F_{n+1}^2 F_{n+2}} \\
= &\frac{(-1)^{n+1}F_{2n+1}}{F_{n-1}F_n^2 F_{n+1}^2 F_{n+2}}.
\end{aligned}$$

(1) If n is even, and $n \geq 2$, then

$$\frac{1}{F_{n-1}F_n} - \frac{1}{F_n^2} - \frac{1}{F_{n+1}^2} - \frac{1}{F_{n+1}F_{n+2}} < 0.$$

Therefore,

$$\frac{1}{F_{n-1}F_n} < \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+1}F_{n+2}}.$$

Using this inequality repeatedly, we have

$$\begin{aligned}
\frac{1}{F_{n-1}F_n} &< \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+1}F_{n+2}} \\
&< \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \left(\frac{1}{F_{n+2}^2} + \frac{1}{F_{n+3}^2} + \frac{1}{F_{n+3}F_{n+4}}\right) \\
&< \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} + \frac{1}{F_{n+3}^2} + \left(\frac{1}{F_{n+4}^2} + \frac{1}{F_{n+5}^2} + \frac{1}{F_{n+5}F_{n+6}}\right) \\
&< \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} + \frac{1}{F_{n+3}^2} + \frac{1}{F_{n+4}^2} + \frac{1}{F_{n+5}^2} + \cdots \\
&= \sum_{k=n}^{\infty} \frac{1}{F_k^2}.
\end{aligned}$$

We have

$$\sum_{k=n}^{\infty} \frac{F_{n-1}F_n}{F_k^2} > 1.$$

(2) If n is odd, and $n \geq 3$, the inequality is similarly obtained. If $n = 1$, then the inequality is easily verified. □

Lemma 4. *For $n \geq 1$,*

(1) $\displaystyle\sum_{k=n}^{\infty} \frac{F_{n-1}F_n - 1}{F_k^2} < 1.$

(2) $\displaystyle\sum_{k=n}^{\infty} \frac{F_{n-1}F_n + 1}{F_k^2} > 1.$

Proof. (1) For $n \geq 2$,

$$\begin{aligned}
&\frac{1}{F_{n-1}F_n - 1} - \frac{1}{F_n^2} - \frac{1}{F_nF_{n+1} - 1} \\
&= \frac{F_nF_{n+1} - F_{n-1}F_n}{(F_{n-1}F_n - 1)(F_nF_{n+1} - 1)} - \frac{1}{F_n^2} \\
&= \frac{F_n^2}{(F_{n-1}F_n - 1)(F_nF_{n+1} - 1)} - \frac{1}{F_n^2} \\
&= \frac{F_n^4 - (F_{n-1}F_n - 1)(F_nF_{n+1} - 1)}{F_n^2(F_{n-1}F_n - 1)(F_nF_{n+1} - 1)}.
\end{aligned}$$

The numerator of RHS is,

$$\begin{aligned}
&F_n^4 - F_{n-1}F_n^2F_{n+1} + F_{n-1}F_n + F_nF_{n+1} - 1\\
=\ &F_n^4 - F_n^2(F_n^2 + (-1)^n) + F_n(F_{n-1} + F_{n+1}) - 1\\
=\ &-(-1)^nF_n^2 + F_n(F_{n-1} + F_{n+1}) - 1\\
>\ &-F_n^2 + F_n(F_{n-1} + F_{n+1}) - 1\\
=\ &F_n(-F_n + F_{n-1} + F_{n+1}) - 1\\
=\ &2F_nF_{n-1} - 1 \geq 1 > 0
\end{aligned}$$

(because $n \geq 2$). Thus we have

$$\frac{1}{F_{n-1}F_n - 1} - \frac{1}{F_n^2} - \frac{1}{F_nF_{n+1} - 1} > 0.$$

Therefore,

$$\frac{1}{F_{n-1}F_n - 1} > \frac{1}{F_n^2} + \frac{1}{F_nF_{n+1} - 1}.$$

Just proceed as above, and we have

$$\begin{aligned}
\frac{1}{F_{n-1}F_n - 1} &> \frac{1}{F_n^2} + \frac{1}{F_nF_{n+1} - 1}\\
&> \frac{1}{F_n^2} + \left(\frac{1}{F_{n+1}^2} + \frac{1}{F_{n+1}F_{n+2} - 1}\right)\\
&> \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \left(\frac{1}{F_{n+2}^2} + \frac{1}{F_{n+2}F_{n+3} - 1}\right)\\
&> \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} + \left(\frac{1}{F_{n+3}^2} + \frac{1}{F_{n+3}F_{n+4} - 1}\right)\\
&> \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} + \frac{1}{F_{n+3}^2} + \frac{1}{F_{n+4}^2} + \cdots\\
&= \sum_{k=n}^{\infty} \frac{1}{F_k^2}.
\end{aligned}$$

We have

$$\sum_{k=n}^{\infty} \frac{F_{n-1}F_n - 1}{F_k^2} < 1, \text{if } n \geq 2.$$

If $n = 1$, then this inequality clearly holds. Thus we obtain (1). (2) is similarly obtained. □

Proof of Theorem 2

Case 1. n is even, and $n \geq 2$.

Using our Lemmas 3(1) and 4(1), we have

$$\frac{1}{F_{n-1}F_n} < \sum_{k=n}^{\infty} \frac{1}{F_k^2} < \frac{1}{F_{n-1}F_n - 1}.$$

Therefore, we have

$$F_{n-1}F_n - 1 < \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2}\right)^{-1} < F_{n-1}F_n.$$

Therefore,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = F_{n-1}F_n - 1.$$

Case 2. n is odd, and $n \geq 1$.
If $n = 1$,

$$\sum_{k=1}^{\infty} \frac{1}{F_k^2} > \frac{1}{F_1^2} = 1 \text{ implies } 0 < \left(\sum_{k=1}^{\infty} \frac{1}{F_k^2} \right)^{-1} < 1.$$

Therefore,

$$\left\lfloor \left(\sum_{k=1}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = 0 = F_0F_1.$$

If n is odd, and $n \geq 3$, with the use of our Lemmas 3(2) and 4(2), we have

$$\frac{1}{F_{n-1}F_n + 1} < \sum_{k=n}^{\infty} \frac{1}{F_k^2} < \frac{1}{F_{n-1}F_n},$$

therefore,

$$F_{n-1}F_n < \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} < F_{n-1}F_n + 1.$$

Therefore,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = F_{n-1}F_n.$$

Thus we have proved Theorem 2. □

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Bunkyo University High School, 1191-7, Kami, Ageo-city, Saitama Pref., 362-0001, JAPAN
E-mail address: otsukahideyuki@gmail.com

Prof. emeritus, Tokyo University of Marine Science and Engineering, 3-29-5, Sakuradai, Nerimaku, Tokyo, 176-0002, JAPAN
E-mail address: shigeru-nn@jcom.home.ne.jp