ON MELHAM'S SUM

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Abstract. The sum $L_1 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$ was first considered by Melham. He noticed that for small m it could be expressed as a polynomial in F_{2n+1}. In this paper we give an explicit expansion for Melham's sum as a polynomial in F_{2n+1}.

1. Introduction

Clary and Hemenway [1] began with the result

$$\sum_{k=1}^{n} F_{2k}^3 = \frac{1}{4}(F_{2n+1}^3 - 3F_{2n+1} + 2).$$

Based on the parity of n they were able to express the sum $\sum_{k=1}^{n} F_{2k}^3$ as a product of Fibonacci and Lucas numbers. This prompted Melham to examine the sum $\sum_{k=1}^{n} F_{2k}^{2m+1}$ for $m = 0, 1, 2, 3, 4$. In each case he noticed that the relevant sum could be expressed as a polynomial in F_{2n+1}. For example $m = 2$ yields

$$L_1 L_3 L_5 \sum_{k=1}^{n} F_{2k}^5 = 4F_{2n+1}^5 - 15F_{2n+1}^3 + 25F_{2n+1} - 14. \tag{1}$$

In private communication with Curtis Cooper [2], Melham suggested that it would be interesting to discover an explicit expansion for $L_1 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$ as a polynomial in F_{2n+1}. Wiemann and Cooper [4] analyzed the constant in this expansion and managed to prove a divisibility result. The purpose of this paper is to give the expansion first suggested by Melham.

2. The Main Result

Throughout this paper m and n represent nonnegative integers. We require the following result for the proof of the first lemma.

$$F_{m+n} = F_m L_n - (-1)^n F_{m-n}. \tag{2}$$

The following lemma was proved by Melham with the use of Binet forms. For the sake of completeness, and by way of contrast, we prove it by induction.

Lemma 1. *If m is an odd integer then*

$$L_m \sum_{k=1}^{n} F_{2mk} = F_{m(2n+1)} - F_m.$$

Proof. The proof is by mathematical induction on n. For $n = 1$, by the multiplication formula (2), we have

$$F_{2m+m} = L_m F_{2m} - (-1)^m F_{2m-m}.$$

Therefore, $L_m F_{2m} = F_{3m} - F_m$.

For $n+1$, we have

$$L_m \sum_{k=1}^{n+1} F_{2mk} = L_m \left(\sum_{k=1}^{n} F_{2mk} + F_{2m(n+1)} \right) = F_{m(2n+1)} - F_m + F_{2m(n+1)} L_m.$$

Again by (2)

$$F_{2m(n+1)+m} = F_{2m(n+1)} L_m - (-1)^m F_{2m(n+1)-m}.$$

Therefore we obtain for $n+1$

$$L_m \sum_{k=1}^{n+1} F_{2mk} = F_{m(2n+3)} - F_m.$$

□

Lemma 2.

$$F_n^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j(n+1)} \binom{2m+1}{j} F_{(2m+1-2j)n}.$$

Proof. Let α and β be the roots of the quadratic equation $x^2 - x - 1 = 0$, so $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Using the Binet formula for Fibonacci numbers, we have

$$\begin{aligned}
F_n^{2m+1} &= \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^{2m+1} \\
&= \frac{1}{(\alpha-\beta)^{2m+1}} \sum_{j=0}^{2m+1} (-1)^{j+1} \binom{2m+1}{j} \alpha^{jn} \beta^{(2m+1-j)n} \\
&= \frac{1}{5^m(\alpha-\beta)} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} (\alpha^{(2m+1-j)n} \beta^{jn} - \alpha^{jn} \beta^{(2m+1-j)n}) \\
&= \frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \alpha^{jn} \beta^{jn} \left(\frac{\alpha^{(2m+1-2j)n} - \beta^{(2m+1-2j)n}}{\alpha - \beta} \right) \\
&= \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j(n+1)} \binom{2m+1}{j} F_{(2m+1-2j)n}.
\end{aligned}$$

□

We require the following result of Jennings [3].

Lemma 3.

$$F_{(2m+1)n} = \sum_{i=0}^{m} (-1)^{n(m+i)} 5^i \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} F_n^{2i+1}.$$

Our main result will follow from the following theorem.

Theorem 1.

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} \frac{(-1)^j}{L_{2m+1-2j}} \binom{2m+1}{j} (F_{(2m+1-2j)(2n+1)} - F_{2m+1-2j}).$$

Proof. In Lemma 2 replace n by $2k$ and sum from $k = 1$ to n. This gives

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \sum_{k=1}^{n} F_{(2m+1-2j)2k}.$$

Theorem 1 now follows from Lemma 1. □

Our main result follows.

Theorem 2. *The expansion of* $\sum_{k=1}^{n} F_{2k}^{2m+1}$ *in powers of* F_{2n+1} *is given by*

$$\sum_{i=0}^{m} F_{2n+1}^{2i+1} \sum_{j=0}^{m-i} \frac{(-1)^{m+i} 5^{i-m} (2m-2j+1)}{L_{2m+1-2j}(m-j+i+1)} \binom{2m+1}{j} \binom{m-j+i+1}{2i+1}$$

$$+ \sum_{j=0}^{m} \frac{(-1)^{j+P} 5^{-m} F_{2m+1-2j}}{L_{2m+1-2j}} \binom{2m+1}{j}.$$

Proof. Using Lemma 3 we obtain an expansion for $F_{(2m+1-2j)(2n+1)}$ and substitute this into the result stated in Theorem 1. This shows that $\sum_{k=1}^{n} F_{2k}^{2m+1}$ is equal to

$$\sum_{j=0}^{m} \sum_{i=0}^{m-j} \frac{(-1)^{m+i} 5^{i-m} (2m-2j+1)}{L_{2m+1-2j}(m-j+i+1)} \binom{2m+1}{j} \binom{m-j+i+1}{2i+1} F_{2n+1}^{2i+1}$$

$$+ \sum_{j=0}^{m} \frac{(-1)^{j+1} 5^{-m} F_{2m+1-2j}}{L_{2m+1-2j}} \binom{2m+1}{j}.$$

Finally we reverse the order of summation in the double sum, and this establishes Theorem 2. □

The reader can readily verify that (1) follows from Theorem 2. As a second example we have, for $m = 3$,

$$L_1 L_3 L_5 L_7 \sum_{k=1}^{n} F_{2k}^{7} = 44F_{2n+1}^{7} - 224F_{2n+1}^{5} + 455F_{2n+1}^{3} - 553F_{2n+1} + 278. \qquad (3)$$

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References

[1] S. Clary and P. D. Hemenway, *On Sums of Cubes of Fibonacci Numbers*, Applications of Fibonacci Numbers, **5** (1993), 123–136.
[2] C. Cooper, private communication.
[3] D. Jennings, *Some Polynomial Identities for the Fibonacci and Lucas Numbers*, The Fibonacci Quarterly, **31.2** (1993), 134–137.
[4] M. Wiemann and C. Cooper, *Divisibility of an F-L Type Convolution*, Applications of Fibonacci Numbers, **9** (2004), 267–287.

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