FIBONACCI-RIESEL AND FIBONACCI-SIERPIŃSKI NUMBERS

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Abstract. Here, we prove that there are infinitely many Fibonacci numbers which are Riesel numbers. We also show that there are infinitely many Fibonacci numbers which are Sierpiński numbers.

1. Introduction

In 1960, W. Sierpiński (see [4]) showed that there are infinitely many odd positive integers k such that $2^n k + 1$ is composite for all n. In 1962, J. Selfridge showed that $k = 78557$ is a Sierpiński number. This is now believed to be the smallest Sierpiński number. As of the writing of this paper, there are only 6 odd candidates k smaller than 78557 whose status with respect to being Sierpiński or not remains to be decided (see [3]).

In a similar vein, a Riesel number is an odd positive integer k such that $2^n k - 1$ is composite for all nonnegative integers n. They were first investigated by H. Riesel [2] in 1956, four years before Sierpiński's paper. There are infinitely many such and it is believed that the smallest Riesel number is 509203.

Typically, the way to find Sierpiński or Riesel numbers is the following. Assume that $(a_i, b_i, p_i)_{i=1}^{i=t}$ are triples of positive integers with the following properties:

(i) for each integer n there exists $i \in \{1, \ldots, t\}$ such that $n \equiv a_i \pmod{b_i}$;

(ii) $p_1, \ldots, p_t$ are distinct prime numbers such that $p_i \mid 2^{b_i} - 1$ for all $i = 1, \ldots, t$.

Then one creates Sierpiński numbers k by imposing that

$$2^{a_i} k \equiv -1 \pmod{p_i} \qquad \text{for } i = 1, \ldots, t. \tag{1.1}$$

Since the primes p_i are all odd for $i = 1, \ldots, t$, it follows that for each i, the above congruence (1.1) is solvable and puts k into a certain arithmetic progression modulo p_i. The fact that the congruences (1.1) are simultaneously solvable for all $i = 1, \ldots, t$ follows from the fact that the primes $p_1, \ldots, p_t$ are distinct via the Chinese Remainder Lemma. Every odd positive integer k in the resulting arithmetic progression has the property that $2^n k + 1$ is always a multiple of one of the numbers p_i for $i = 1, \ldots, t$, and if $k > \max\{p_i : i = 1, \ldots, t\}$, it follows that $2^n k + 1$ can never be a prime.

Similarly, one creates Riesel numbers by taking odd numbers k such that $2^{a_i} k \equiv 1 \pmod{p_i}$. Again, all such numbers belong to a fixed arithmetic progression modulo $2p_1 \cdots p_t$, and all but finitely many of them will have the property that $2^n k - 1$ is divisible by one of the primes $p_1, \ldots, p_t$ for all nonnegative integers n, and exceeds the largest one of them, therefore all such k are Riesel numbers.

Let $(F_n)_{n \geq 0}$ be the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. In [1], Luca and Stănică showed that there exist infinitely many

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positive integers n such that $F_n \neq p^a + q^b$ for prime powers p^a and q^b. Here, we use a similar method and prove the following result.

Theorem 1.1.
(1) *There are infinitely many Fibonacci numbers which are Riesel numbers.*
(2) *There are infinitely many Fibonacci numbers which are Sierpiński numbers.*

2. The Proof of Theorem 1.1

Let $(a_i, b_i, p_i)_{i=1}^{i=t}$ be a set of t triples satisfying conditions (i) and (ii) above. Let k be a Sierpiński or Riesel number constructed using the above system by the procedure indicated at the beginning of this note. Then $k \equiv \varepsilon 2^{-a_i} \pmod{p_i}$ for all $i = 1, \ldots, t$, where $\varepsilon = -1$ or 1 according to whether k is Sierpiński or Riesel, respectively. For a positive integer m let $h(m)$ be the period of the Fibonacci sequence modulo m, and for an integer x let

$$\mathcal{A}(x, m) = \{0 \le y \le h(m) - 1 : F_y \equiv x \pmod{m}\} + h(m)\mathbb{Z}.$$

That is, $\mathcal{A}(x, m)$ is the set of all congruence classes y modulo $h(m)$ such that $F_y \equiv x \pmod{m}$. In order for the number k above to be a Fibonacci number it is necessary that the congruence $k \equiv F_{y_i} \pmod{p_i}$ has a solution y_i for each $i = 1, \ldots, t$. But note that the set of solutions y_i of the above congruence is precisely

$$\mathcal{A}(\varepsilon 2^{-a_i}, p_i).$$

Thus, in order for some n to exist such that k can be taken to be of the form F_n, it is necessary and sufficient that

$$\mathcal{A} = \bigcap_{i=1}^{t} \mathcal{A}(\varepsilon 2^{-a_i}, p_i) \neq \emptyset.$$

Finally, since F_n must also be odd, the above nonempty set of arithmetic progressions should contain a number which is not a multiple of 3.

Armed with these facts, we are ready to prove our theorem.

2.1. The Riesel Case.

We start with the Riesel case, since for it our proof is simpler.

We take $t = 7$ and

$$(a_i, b_i)_{i=1}^{i=7} = ((0,2), (0,3), (1,4), (11,12), (7,36), (19,36), (31,36),$$

and we get that (i) above is fulfilled. Indeed, every integer is congruent either to $0 \pmod{2}$, or to $0 \pmod{3}$, or to $1 \pmod{4}$, or to $11 \pmod{12}$, or to $7 \pmod{12}$, and in this last case it is congruent to either 7, 19, or 31 modulo 36. We now take $(p_1, \ldots, p_7) = (3, 7, 5, 13, 73, 37, 19)$ and note that condition (ii) above is fulfilled. It is easy to check that $(h(p_1), \ldots, h(p_7)) = (8, 16, 20, 28, 148, 76, 18)$. Furthermore, it is also easy to see that

$$\begin{array}{rcl}
\mathcal{A}(2^{-a_1}, p_1) & = & \mathcal{A}(1,3) = \{1,2,7\} \pmod{8}, \\
\mathcal{A}(2^{-a_2}, p_2) & = & \mathcal{A}(1,7) = \{1,2,6,15\} \pmod{16}, \\
\mathcal{A}(2^{-a_3}, p_3) & = & \mathcal{A}(3,5) = \{4,6,7,13\} \pmod{20}, \\
\mathcal{A}(2^{-a_4}, p_1) & = & \mathcal{A}(2,13) = \{3,25\} \pmod{28}, \\
\mathcal{A}(2^{-a_5}, p_5) & = & \mathcal{A}(4,73) = \{53,95\} \pmod{148}, \\
\mathcal{A}(2^{-a_6}, p_6) & = & \mathcal{A}(18,37) = \{10,15,28,61\} \pmod{76}, \\
\mathcal{A}(2^{-a_7}, p_7) & = & \mathcal{A}(13,19) = \{7,11\} \pmod{18}.
\end{array}$$

One checks now that $n \equiv 1807873 \pmod{3543120}$ is odd and belongs to all $\mathcal{A}(2^{-a_i}, p_i)$ for $i = 1, \ldots, 7$. Thus, for such n, the number F_n is a Riesel number.

2.2. **The Sierpiński Case.** For the Sierpiński case, we take $t = 9$ and

$$\begin{aligned}(a_i, b_i)_{i=1}^{i=9} &= (1,2),(2,4),(4,8),(8,16),(16,32),(32,64),(0,192),\\ &\quad (64,192),(128,192).\end{aligned}$$

Indeed, every number is either congruent to 1 (mod 2), or 2 (mod 4), or 4 (mod 8), or 8 (mod 16), or 16 (mod 32), or 32 (mod 64), or 0 (mod 64), and in this last case it is congruent to either 0, or 64, or 128 modulo 192. We take

$$(p_i)_{i=1}^{9} = (3, 5, 17, 257, 65537, 641, 13, 673, 193),$$

and observe that

$$(h(p_i))_{i=1}^{i=9} = (8, 20, 36, 516, 14564, 640, 28, 1348, 388).$$

Furthermore,

$$\begin{aligned}
\mathcal{A}(-2^{-a_1}, p_1) &= \mathcal{A}(1,3) = \{1,2,7\} \pmod{8},\\
\mathcal{A}(-2^{-a_2}, p_2) &= \mathcal{A}(1,5) = \{1,2,8,19\} \pmod{20},\\
\mathcal{A}(-2^{-a_3}, p_3) &= \mathcal{A}(1,17) = \{1,2,16,35\} \pmod{36},\\
\mathcal{A}(-2^{-a_4}, p_4) &= \mathcal{A}(1,257) = \{1,2,256,515\} \pmod{516},\\
\mathcal{A}(-2^{-a_5}, p_5) &= \mathcal{A}(1,65537) = \{1,2,7280,14563\} \pmod{14564},\\
\mathcal{A}(-2^{-a_6}, p_6) &= \mathcal{A}(1,641) = \{1,2,318,639\} \pmod{640},\\
\mathcal{A}(-2^{-a_7}, p_7) &= \mathcal{A}(12,13) = \{13,15,16,26\} \pmod{28},\\
\mathcal{A}(-2^{-a_8}, p_8) &= \mathcal{A}(256,673) = \{53,1295\} \pmod{1348},\\
\mathcal{A}(-2^{-a_9}, p_9) &= \mathcal{A}(109,193) = \{109,279\} \pmod{388}.
\end{aligned}$$

The least common multiple of the above moduli is

$$M = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 43 \cdot 97 \cdot 331 \cdot 337 = 206353240410240.$$

Furthermore, if $n \equiv 20808199653121 \pmod{M}$, then $n \in \bigcap_{i=1}^{9} \mathcal{A}(-2^{-a_i}, p_i)$. This shows that there exist infinitely many Fibonacci numbers which are also Sierpiński numbers.

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