

RECURRENCE RELATIONS FOR NÖRLUND NUMBERS AND BERNOULLI NUMBERS OF THE SECOND KIND

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ABSTRACT. Known recurrence relations for the Nörlund numbers and the Bernoulli numbers of the second kind are generalized to infinite classes of recurrence relations. The Stirling numbers of the first kind occur in all these recurrences. Some of the relations are of arbitrary length and contain explicit formulas as special cases. A convolution formula for Stirling numbers of the first kind is also obtained.

1. INTRODUCTION

A large number of recurrence relations of various different types are known for the Bernoulli numbers; see, for instance, the introduction of [2] for a historical perspective and for references. In this paper we deal with recurrence relations for two related sequences of rational numbers, namely the Nörlund numbers and the Bernoulli numbers of the second kind.

To define the Nörlund numbers, we first consider the Bernoulli numbers of higher order, which were studied by Nörlund [9] in connection with deeper investigations in the calculus of finite differences. The Bernoulli numbers of order r can be defined by the generating function

$$\left(\frac{x}{e^x - 1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{x^n}{n!} \quad (|x| < 2\pi); \quad (1.1)$$

[9, p. 145]. Then $B_n^{(1)} = B_n$, the ordinary Bernoulli numbers. Although usually the order r is assumed to be a positive integer, it can be seen that $B_n^{(r)}$ is a polynomial in r of degree n , so that r can actually be extended to any real or complex value. We will not be concerned with this, but we remark that Bernoulli numbers of negative orders are just Stirling numbers of the second kind up to a binomial factor.

We now proceed to the sequence of numbers $B_n^{(n)}$, called *Nörlund numbers* by Howard [5], i.e., higher-order Bernoulli numbers with the order equal to the index. Nörlund found the exponential generating function

$$\frac{x}{(1+x)\log(1+x)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{x^n}{n!} \quad (|x| < 1), \quad (1.2)$$

and the simple recurrence relation,

$$\sum_{j=0}^n \frac{(-1)^j}{n-j+1} \frac{B_j^{(j)}}{j!} = 1, \quad B_0^{(0)} = 1 \quad (1.3)$$

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[9, p. 150]. Nörlund also listed the first few values $1, -\frac{1}{2}, \frac{5}{6}, -\frac{9}{4}, \frac{251}{30}, -\frac{475}{12}$, and Howard [5] derived numerous divisibility and congruence properties. Among the uses of Nörlund numbers are the remarkable expansions [9, p. 244],

$$\log\left(1 + \frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n)}}{(x+1)(x+2)\dots(x+n)},$$

$$\Psi(x) = \log x - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n B_n^{(n)}}{n} \frac{1}{(x+1)(x+2)\dots(x+n)},$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is Euler's Ψ -function, the logarithmic derivative of the Gamma function. Both expansions are valid for $\Re(x) > 0$.

The *Bernoulli numbers of the second kind* were introduced and studied by Jordan [6, p. 265 ff.], also in connection with the calculus of finite differences. These numbers b_n can be defined by way of the generating function

$$\frac{x}{\log(x+1)} = \sum_{n=0}^{\infty} b_n x^n \quad (|x| < 1); \quad (1.4)$$

[6, p. 279], or [5]. If we compare (1.4) with (1.2), as was done in [5], we immediately obtain the relations

$$B_n^{(n)} = n! \sum_{j=0}^n (-1)^{n-j} b_j, \quad b_n = \frac{1}{n!} B_n^{(n)} + \frac{1}{(n-1)!} B_{n-1}^{(n-1)} \quad (n \geq 1). \quad (1.5)$$

The Bernoulli numbers of the second kind satisfy the simple recurrence relation

$$\sum_{j=0}^n \frac{(-1)^j}{n-j+1} b_j = 0 \quad (n \geq 1), \quad (1.6)$$

with $b_0 = 1$ [6, p. 266], but note the small misprint there), and the first few values can be easily computed as $1, \frac{1}{2}, -\frac{1}{12}, \frac{1}{24}, -\frac{19}{720}, \frac{3}{160}$. One application of the numbers b_n lies in the integration of functions expanded in Newton series. This leads, for example, to the interesting identities [6, pp. 277, 280]

$$1 - \log 2 = \sum_{n=1}^{\infty} \frac{|b_n|}{n+1}, \quad \gamma = \sum_{n=1}^{\infty} \frac{|b_n|}{n},$$

where γ is the Euler constant.

It is the purpose of this paper to extend or generalize the recurrence relations (1.3) and (1.6). In both cases the Stirling numbers of the first kind play an important role; we summarize some of their properties in Section 2. In Section 3 we deal with the Nörlund numbers, and in Section 4 with the Bernoulli numbers of the second kind. The final section contains some additional remarks, including a convolution formula for Stirling numbers of the first kind and relations with the Cauchy numbers.

2. STIRLING NUMBERS OF THE FIRST KIND

The *Stirling numbers* of the first and the second kind belong to the most basic objects in combinatorics, with important applications in other areas of mathematics. In this paper

only the Stirling numbers *of the first kind*, $s(n, k)$, will occur. They can be defined by the generating function

$$x(x-1)\dots(x-n+1) = \sum_{k=0}^n s(n, k)x^k. \quad (2.1)$$

This means that they are the coefficients connecting the two most fundamental bases of the vector space of single-variable polynomials (while the inverse transformation between these two bases is given by the Stirling numbers of the second kind).

For a combinatorial interpretation see [4]. Among their many properties the following will be used in this paper:

$$s(0, 0) = s(n, n) = 1, \quad s(n, 0) = 0 \quad \text{for } n \geq 1, \quad (2.2)$$

$$s(n, 1) = (-1)^{n-1}(n-1)!, \quad (2.3)$$

$$s(n, 2) = (-1)^n(n-1)!H_{n-1}, \quad (2.4)$$

where $H_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$ is the k th harmonic number;

$$s(n, n-2) = \frac{(n-1)n(3n+2)(n+1)}{24}, \quad (2.5)$$

$$s(n, n-1) = -\binom{n}{2}, \quad (2.6)$$

in addition to $s(n, k) = 0$ for $k < 0$ and for $k > n$. The most basic recurrence relation is

$$s(n+1, k) = s(n, k-1) - ns(n, k), \quad 1 \leq k \leq n. \quad (2.7)$$

While the generating function (2.1) has a fixed n and index of summation k , it is the other way for the (exponential) generating function

$$(\log(x+1))^k = \sum_{n=k}^{\infty} k!s(n, k)\frac{x^n}{n!} \quad (|x| < 1), \quad (2.8)$$

which will be essential in the remainder of this paper.

These and numerous other properties can be found, e.g., in the books [1, Ch. 24], [3, 4], or in the on-line resources [11] or [10, A008277]. Although there are some advantages to the bracket notation used in [4] (see also [7]) we use here the main competing notation $s(n, k)$.

3. NÖRLUND NUMBERS

It is the purpose of this brief section to derive a general recurrence relation for the Nörlund numbers $B_n^{(n)}$. The Stirling numbers of the first kind enter via their exponential generating function (2.8), which gives

$$\begin{aligned} \left(1 + \frac{1}{x}\right)(\log(x+1))^k &= \sum_{n=k}^{\infty} k!s(n, k)\frac{x^n}{n!} + \sum_{n=k}^{\infty} k!s(n, k)\frac{x^{n-1}}{n!} \\ &= \sum_{n=k-1}^{\infty} k!\left(s(n, k) + \frac{1}{n+1}s(n+1, k)\right)\frac{x^n}{n!}. \end{aligned}$$

If we multiply the last power series with the right-hand side of (1.2), we obtain

$$(\log(x+1))^{k-1} = \sum_{n=k-1}^{\infty} \left(\sum_{j=0}^{n-k+1} \binom{n}{j} k! \left(s(n-j, k) + \frac{s(n-j+1, k)}{n-j+1} \right) B_j^{(j)} \right) \frac{x^n}{n!}.$$

Equating coefficients of x^n in this power series and in (2.8), we get

$$\sum_{j=0}^{n-k+1} \binom{n}{j} \left(s(n-j, k) + \frac{1}{n-j+1} s(n-j+1, k) \right) B_j^{(j)} = \frac{1}{k} s(n, k-1). \quad (3.1)$$

We use (2.7) in the form $(n-j)s(n-j, k) + s(n-j+1, k) = s(n-j, k-1)$, add $s(n-j, k)$ to both sides, and divide by $n-j+1$; then the term in large parentheses in (3.1) becomes $(s(n-j, k-1) + s(n-j, k))/(n-j+1)$, which finally gives the following result.

Theorem 3.1. *For any $k \geq 1$ and $n \geq k-1$ we have*

$$\sum_{j=0}^{n-k+1} \binom{n+1}{j} (s(n-j, k-1) + s(n-j, k)) B_j^{(j)} = \frac{n+1}{k} s(n, k-1). \quad (3.2)$$

We now state two special cases, for $k=1$ and $k=2$. They are easily derived from (3.2) by using (2.2)–(2.4).

Corollary 3.1. *For any $n \geq 1$ we have*

$$\sum_{j=0}^{n-1} \frac{(-1)^j}{(n-j)(n-j+1)} \frac{B_j^{(j)}}{j!} + \frac{(-1)^{n-1}}{n!} B_n^{(n)} = 0, \quad (3.3)$$

and for $n \geq 0$,

$$\sum_{j=0}^n (-1)^{j+1} \frac{H_{n-j} - 1}{(n-j+1)(n-j+2)} \frac{B_j^{(j)}}{j!} = \frac{1}{2n+2}, \quad (3.4)$$

with H_k as in (2.4).

The recurrence relation (3.3) can also be obtained independently from (1.3) by subtracting (1.3) for n from the same identity for $n-1$. Also, (3.3) was earlier obtained by Howard [5] who used it to show that Nörlund numbers have alternating signs. The relation (3.3) is valid, as stated, only for $n \geq 1$ since $s(n, 0) = \delta_{n0}$; see (2.2). Also note that $H_0 = 0$ as an empty sum. See Section 5 below for some further remarks on recurrences for Nörlund numbers.

4. BERNOULLI NUMBERS OF THE SECOND KIND

We begin by stating the desired generalization of the recurrence relation (1.6).

Theorem 4.1. *For any $k \geq 1$ and $n \geq k$ we have*

$$\sum_{j=0}^{n-k} \frac{s(n-j, k)}{(n-j)!} b_j = \frac{s(n-1, k-1)}{(n-1)!k}. \quad (4.1)$$

We state the first two cases (for $k=1, 2$) separately. The next two identities follow immediately from (4.1), with (2.3), resp. (2.4).

Corollary 4.1. *For any $n \geq 2$ we have*

$$\sum_{j=0}^{n-1} \frac{(-1)^j}{n-j} b_j = 0; \tag{4.2}$$

$$\sum_{j=0}^{n-2} \frac{(-1)^j}{n-j} H_{n-1-j} b_j = \frac{1}{2(n-1)}, \tag{4.3}$$

where H_k is the k th harmonic number.

The identity (4.2) is the same as (1.6), but (4.3) and the general case (4.1) appear to be new. Theorem 4.1 would be easy to prove by considering the product of the generating functions (1.4) and (2.8). However, it is not much more difficult to derive a substantially more general (though complicated) result. To do this, we introduce the expression

$$\tilde{s}(n, k, m) := \sum_{r=0}^m \binom{m}{r} \frac{s(n-r, k+m)}{(n-r)!}, \tag{4.4}$$

which enters through the following product of power series: With (2.8) we get

$$\begin{aligned} & (x+1)^m (\log(x+1))^{k+m} \\ &= \left(\sum_{j=0}^{\infty} \binom{m}{j} x^j \right) \left(\sum_{j=k+m}^{\infty} \frac{(k+m)!}{j!} s(j, k+m) x^j \right) \\ &= (k+m)! \sum_{n=k+m}^{\infty} \left(\sum_{j=0}^{n-k-m} \binom{m}{j} \frac{s(n-j, k+m)}{(n-j)!} \right) x^n, \end{aligned}$$

where the inner summation can be taken to m instead of $n-k-m$. So with (4.4) we have

$$(x+1)^m (\log(x+1))^{k+m} = (k+m)! \sum_{n=k+m}^{\infty} \tilde{s}(n, k, m) x^n. \tag{4.5}$$

We also need the following lemma, which is easy to prove by induction on m , where the identity (2.7) will be essential.

Lemma 4.1. *For any $m \geq 0$ we have*

$$\frac{d^m}{dx^m} \frac{1}{\log(x+1)} = \frac{1}{(x+1)^m} \sum_{j=0}^m (-1)^j j! \frac{s(m, j)}{(\log(x+1))^{j+1}}. \tag{4.6}$$

We are now ready to prove the following generalization of Theorem 4.1.

Theorem 4.2. *For any $m \geq 0, k \geq 1$, and $n \geq k+m$ we have*

$$\begin{aligned} & \sum_{j=0}^{n-k-m} \binom{m+j}{m} \tilde{s}(n-j, k, m) b_{m+1+j} + (-1)^m \tilde{s}(n+m+1, k, m) \\ &= \frac{1}{n! m! (k+m)} \sum_{j=0}^m (-1)^{m-j} \frac{s(m, m-j) s(n, k-1+j)}{\binom{k+m-1}{k+j-1}}, \end{aligned} \tag{4.7}$$

with \tilde{s} as defined in (4.4).

Proof. From (1.4) we immediately obtain, for $m \geq 0$,

$$\frac{d^m}{dx^m} \frac{1}{\log(x+1)} = (-1)^m \frac{m!}{x^{m+1}} + \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} b_{n+m+1} x^n. \quad (4.8)$$

The identities (4.8) and (4.5) multiplied together give

$$\begin{aligned} B(x) &:= (x+1)^m (\log(x+1))^{k+m} \frac{d^m}{dx^m} \frac{1}{\log(x+1)} \\ &= \sum_{n=k+m}^{\infty} \left(\sum_{j=0}^{n-k-m} (k+m)! \tilde{s}(n-j, k, m) \frac{(m+j)!}{j!} b_{m+1+j} \right) x^n \\ &\quad + \sum_{n=k+m}^{\infty} (-1)^m m! (k+m)! \tilde{s}(n, k, m) x^{n-m-1} \\ &= \sum_{n=k-1}^{\infty} m! (k+m)! \\ &\quad \times \left(\sum_{j=0}^{n-k-m} \binom{m+j}{m} \tilde{s}(n-j, k, m) b_{m+1+j} + (-1)^m \tilde{s}(n+m+1, k, m) \right) x^n. \end{aligned}$$

(Here, as elsewhere, a sum will be empty and thus vanish when the upper limit of summation is less than the lower limit). On the other hand, multiplying the left-hand side of (4.5) with (4.6) we get, upon using (2.8) again,

$$\begin{aligned} B(x) &= \sum_{j=0}^m (-1)^j j! s(m, j) (\log(x+1))^{k+m-j-1} \\ &= \sum_{j=0}^m (-1)^{m-j} (m-j)! s(m, m-j) (\log(x+1))^{k+j-1} \\ &= \sum_{j=0}^m (-1)^{m-j} (m-j)! s(m, m-j) \sum_{n=k+j-1}^{\infty} (k+j-1)! s(n, k+j-1) \frac{x^n}{n!} \\ &= \sum_{n=k-1}^{\infty} \left(\sum_{j=0}^m (-1)^{m-j} (m-j)! s(m, m-j) (k+j-1)! s(n, k+j-1) \right) \frac{x^n}{n!}. \end{aligned}$$

We obtain (4.7) by equating coefficients of x^n in both evaluations of $B(x)$. \square

Note that the above proof is valid also for $n < k+m$; we will deal with this in the next section.

Proof of Theorem 4.1. We set $m=0$ and use the fact that from (4.4) we immediately get $\tilde{s}(n, k, 0) = s(n, k)/n!$. Then we have, by (2.2),

$$\sum_{j=0}^{n-k} \frac{s(n-j, k)}{(n-j)!} b_{j+1} + \frac{s(n+1, k)}{(n+1)!} = \frac{s(n, k-1)}{n!k}.$$

Finally we change the summation to range from 1 to $n+1-k$, replace $n+1$ by n , and use the fact that $b_0 = 1$, to obtain (4.1). \square

The other extreme case of Theorem 4.2 occurs when $n = k + m$. With (4.4) we immediately get the following result in this case.

Corollary 4.2. *For any $m \geq 0$ and $k \geq 1$ we have*

$$\begin{aligned} (-1)^{m+1}b_{m+1} &= (k+m)! \sum_{r=0}^m \binom{m}{r} \frac{s(k+m+1+r, k+m)}{(k+m+1+r)!} \\ &+ \frac{1}{m!(k+m)} \sum_{j=0}^m (-1)^{j+1} \frac{s(m, m-j)s(k+m, k-1+j)}{\binom{k+m-1}{k+j-1}}. \end{aligned} \tag{4.9}$$

It should be pointed out that this identity is far from the easiest expression of b_n in terms of Stirling numbers. In fact, the simple identity

$$b_n = \frac{1}{n!} \sum_{j=1}^{n+1} \frac{s(n, j)}{j+1}$$

is proved in [6, p. 267], where this expression is also used to derive the relation

$$\sum_{j=1}^{n+1} j!S(n, j)b_j = \frac{1}{n+1},$$

which involves the Stirling numbers of the *second* kind, in contrast to all the other identities in this section.

5. FURTHER REMARKS

1. A notable feature of Theorem 4.2 is the fact that the identity (4.7) is a recurrence relation of variable length. Indeed, we already considered the two extreme cases of “full length”, namely $m = 0$, which led to (4.1), and “length 1”, which gave the expression (4.9). Such formulas also exist for the classical Bernoulli numbers. They were recently studied by the authors in [2], where a historical perspective and references are given.

Recurrence relations of variable length can also be derived for the Nörlund numbers. Given the complexity of such formulas, we only state the following result, without proof.

For all $0 \leq m \leq n$ and all $k \geq 1$ we have

$$\sum_{j=0}^{n-k-m} \frac{\tilde{s}(n-j, k, m)}{j!} B_{m+j-1}^{(m+j-1)} = \frac{1}{n!} \sum_{j=1}^m (-1)^{j-1} \frac{N(n, m, k, j)}{j \binom{k+m}{j}},$$

where \tilde{s} is as defined in (4.4), and

$$N(n, m, k, j) := ns(m-1, j-1)s(n-1, k+m-j) + (m-1)s(m-1, j)s(n, k+m-j).$$

The method of proof is similar to that of Theorem 4.2. When $m = 1$, this reduces to (3.1), and for the other extreme case, $n = k + m$, we would obtain an expression for $B_n^{(n)}$ in terms of the Stirling number of the first kind. Again, such an identity would be more complicated than any of the simpler expressions derived by Howard [5], among them

$$B_n^{(n)} = \frac{1}{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} s(n+1, 2k+1),$$

which has only half the usual number of terms in the summation.

2. If we set $k = 1$ and $n = m$ in (4.7), then the summation on the left-hand side of (4.7) vanishes. This is valid by the proof of Theorem 4.2 which holds for a wider range than just $n \geq k + m$. Thus we get

$$\sum_{j=0}^{m-1} (-1)^j \frac{s(m, j)s(m, m-j)}{\binom{m}{j}} = m!(m+1)! \sum_{r=0}^m \binom{m}{r} \frac{s(2m+1-r, m+1)}{(2m+1-r)!}.$$

This is a convolution identity for Stirling numbers. Such identities have been studied before, and will be the subject of a separate paper.

3. Two special sequences of rational numbers, namely the Cauchy numbers of the first kind, \mathcal{C}_n , and of the second kind, $\hat{\mathcal{C}}_n$, are closely related to the Bernoulli numbers of the second kind and to the Nörlund numbers, respectively. They appear in Exercise 13 in [3, p. 293] and were more recently studied in great detail in [8], but are otherwise not very well known. While a different perspective is given in [8], using the concept of Riordan arrays, many of the results are equivalent to those quoted in this paper, especially in Section 1. A comparison with the definitions in [8] shows that

$$\mathcal{C}_n = n!b_n, \quad \hat{\mathcal{C}}_n = B_n^{(n)}.$$

Therefore our main results could also be written in terms of the Cauchy numbers of both kinds.

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