ON THE DIOPHANTINE EQUATION $x^2 + 2^a \cdot 11^b = y^n$

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ABSTRACT. In this note, we resolve the Diophantine equation $x^2+2^a \cdot 11^b = y^n$ with coprime positive integers x, y and positive integers $n \geq 3$.

1. Introduction

The history of the Diophantine equation

$$x^2 + C = y^n, \;\; x \geq 1, \;\; y \geq 1, \;\; n \geq 3 \tag{1}$$

goes back to the 1850's. In 1850, Lebesque [24] proved that the Diophantine equation (1) has no solutions when $C = 1$. For a fixed value of n, the Diophantine equation (1) is actually a special case of the Diophantine equation $ay^2 + by + c = dx^n$, where a, b, c and d are integers with $a \neq 0$, $b^2 - 4ac \neq 0$ and $d \neq 0$, which has only a finite number of solutions in integers x and y when $n \geq 3$ [22]. Cohn [17] solved the Diophantine equation (1) for most values of C in the range $1 \leq C \leq 100$. In [29], Mignotte and de Weger found all the positive integer solutions (x, y) of the two Diophantine equations $x^2 + 74 = y^5$ and $x^2 + 86 = y^5$, respectively, thus covering some of the cases left over by Cohn. In [14], Bugeaud, Mignotte and Siksek covered the remaining cases.

Variations of the Diophantine equation (1) were also considered by various mathematicians. For theoretical upper bounds for the exponent n we refer to [16] or [21], however, these estimates are based on Baker's theory, so they are huge. In [32], all the positive integer solutions (x, y, n) of the Diophantine equation $x^2 + B^2 = 2y^n$ with $B \in \{3, 4, \ldots, 501\}$ were found under the conditions that $n \geq 3$ and that $\gcd(x, y) = 1$. The equation $x^2 + C = 2y^n$ with C a fixed positive integer and under the similar restrictions $n \geq 3$ and $\gcd(x, y) = 1$ was studied in [2].

Yet a different variant of this problem when C is an arbitrary power of a fixed prime was studied by various authors. In [10], the positive integer solutions (x, y, k, n) of the Diophantine equations $x^2 + 2^k = y^n$ satisfying certain conditions have been found. In [23], Le verified a conjecture of Cohn from [18] proving that all the solutions of the Diophantine equation $x^2 + 2^k = y^n$ in positive integers x, y, k, n with $2 \nmid y$ and $n \geq 3$ are

$$(x, y, k, n) = (5, 3, 1, 3), (7, 3, 5, 4), (11, 5, 2, 3).$$

All the integer solutions (x, y, m, n) of the Diophantine equation $x^2 + 3^m = y^n$ with $n \geq 3$ were found in [9] (for odd m) and in [26] (for even m). For various results on other particular cases of the Diophantine equation $x^2 + p^m = y^n$, where p is a fixed prime, see [4, 6, 7, 8, 27].

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The last variant of the Diophantine equation (1) that we mention is when C is a product of powers of a few fixed primes. For example, all the positive integer solutions of the Diophantine equation (1), again under the assumptions that $n \geq 3$ and x and y are coprime with C of the forms $C = 2^a \cdot 3^b$, $2^a \cdot 5^b$, $5^a \cdot 13^b$, $2^a \cdot 5^b \cdot 13^c$ were found in [3, 20, 25] and [28], respectively. Pink [30] has obtained some results on the case $C = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$. Recently, Bérczes and Pink [12] resolved (1) with $C = p^{2k}$ where $2 \leq p < 100$ prime, $\gcd(x, y) = 1$ and $n \geq 3$. A more exhaustive survey on this type of problem is [5].

Here, we add to the existing literature on this last type of Diophantine equation by studying the Diophantine equation

$$x^2 + 2^a \cdot 11^b = y^n, \quad x \geq 1, \quad y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3, \ a \geq 0, \ b \geq 0, \tag{2}$$

again under the assumptions that $n \geq 3$ and that x and y are coprime. Our result is the following.

Theorem 1. *The only solutions of the Diophantine equation* (2) *are*

$$\begin{aligned}
n &= 3, \quad (x, y, a, b) \in \{(2, 5, 0, 2), (4, 3, 0, 1), (5, 3, 1, 0), (5, 9, 6, 1), \\
&\quad\ (9, 5, 2, 1), (11, 5, 2, 0), (58, 15, 0, 1), (117, 25, 4, 2), (835, 89, 6, 2), \\
&\quad\ (5497, 785, 8, 6), (5805, 323, 1, 2), (6179, 345, 18, 1), (9324, 443, 0, 3), \\
&\quad\ (9959, 465, 10, 3), (404003, 5465, 12, 2)\}; \\
n &= 4, \quad (x, y, a, b) = (7, 3, 5, 0); \\
n &= 5, \quad (x, y, a, b) = (1, 3, 1, 2), (241, 9, 3, 2); \\
n &= 6, \quad (x, y, a, b) = (5, 3, 6, 1), (117, 5, 4, 2); \\
n &= 10, \quad (x, y, a, b) = (241, 3, 3, 2).
\end{aligned}$$

One can deduce from the above result the following corollary.

Corollary 2. *The only integer solutions of the Diophantine equation*

$$x^2 + 11^c = y^n, \quad x \geq 1, \quad y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3, \ c > 0$$

are $(x, y, c, n) = (2, 5, 2, 3), (4, 3, 1, 3), (58, 15, 1, 3), (9324, 443, 3, 3)$.

Several cases of the Diophantine equation (2) have been dealt with previously. For example, for $a = 0$ and odd b, all solutions have been found in [31] by using an elementary method, while for $a = 0$ and even b they appear in [12]. The fact that there are no solutions when $a \geq 3$ and $n \geq 13$ follows from Theorem 1.3 in [11], where the method used was the modular approach à la Wiles' proof of Fermat's Last Theorem. The remaining cases seem to be new.

For the proof, we apply the method used in [3]. We first treat the cases $n = 4$ and $n = 3$. This is done by transforming equation (2) into several elliptic equations written in quartic models and cubic models, respectively, for which we need to determine all their $\{2, 11\}$-integer points. At this stage we note that in [19] Gebel, Hermann, Pethő and Zimmer developed a practical method for computing all S-integral points on elliptic curves. Their method is implemented in MAGMA as a routine under the name `SIntegral Points`. In the last section, we study the remaining cases by using primitive divisors of Lucas sequences. All the computations are done with MAGMA [15] and Cremona's program `mwrank`.

Before starting, we note that since $n \geq 3$, it follows that n is either a multiple of 4 or a multiple of an odd prime p. Furthermore, if $d \mid n$ is such that $d \in \{4, p\}$ with p an odd prime and (x, y, a, b, n) is a solution of our equation (2), then $(x, y^{n/d}, a, b, d)$ is also a solution of our equation satisfying the same restrictions. Thus, we may replace n by d and y by $y^{n/d}$ and from now on assume that $n \in \{4, p\}$. Furthermore, note that when $b = 0$, then our equation reduces to the equation $x^2 + 2^a = y^n$, all whose solutions are already known from [23]. Thus, we shall assume that $b > 0$. Since $11^b \equiv 3, 1 \pmod 8$ according to whether b is odd or even, respectively, it follows by considerations modulo 8 that either $a > 0$, or that x is even, for otherwise with $a = 0$ and x odd we would get that $x^2 + 11^b \equiv 2, 4 \pmod 8$, and this last even number cannot be a perfect power of exponent ≥ 3 of some integer.

2. The Case $n = 4$

Here we have the following result.

Lemma 3. *The Diophantine equation* (2) *has no solution with* $n = 4$ *and* $b > 0$.

Proof. Equation (2) can be written as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4,$$

where A is fourth-power free and defined implicitly by $2^a \cdot 11^b = A \cdot z^4$. Clearly, $A = 2^{a_1} \cdot 11^{b_1}$, where $a_1, b_1 \in \{0, 1, 2, 3\}$. We recall here that if $\mathcal{S}$ is a finite set of prime numbers, then an $\mathcal{S}$-integer is a rational number of the form r/s with coprime integers r and $s > 0$ such that all the prime factors of s are in $\mathcal{S}$. Thus, the problem is reduced to determining all the $\{2, 11\}$-integer points $(U, V) = (y/z, x/z^2)$ on the 16 elliptic curves in quartic models

$$V^2 = U^4 - 2^{a_1} \cdot 11^{b_1},$$

with $a_1, b_1 \in \{0, 1, 2, 3\}$. We use the subroutine `SIntegralLjunggrenPoints` of MAGMA [15] to determine the $\{2, 11\}$-integral points on the above elliptic curves and we only find the following solutions

$$(U, V, a_1, b_1) = (\pm 1, 0, 0, 0,), \ (\pm 3/2, \pm 7/4, 1, 0).$$

They do not lead to solutions of our original equation with $b > 0$. □

3. The Case $n = 3$

Lemma 4. *The only solutions* (x, y, a, b) *to equation* (2) *with* $n = 3$ *and* $b > 0$ *are*

$$(2, 5, 0, 2), (4, 3, 0, 1), (5, 9, 6, 1), (9, 5, 2, 1), (58, 15, 0, 1), (117, 25, 4, 2),$$
$$(835, 89, 6, 2), (5497, 785, 8, 6), (5805, 323, 1, 2), (6179, 345, 18, 1),$$
$$(9324, 443, 0, 3), (9959, 465, 10, 3), (404003, 5465, 12, 2).$$

Proof. We proceed as in the previous section except that now we create cubic models of elliptic equations. Namely, we rewrite equation (2) as

$$\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3,$$

where A is sixth-power free and defined implicitly by $2^a \cdot 11^b = A \cdot z^6$. Certainly, $A = 2^{a_1} \cdot 11^{b_1}$, where $a_1, b_1 \in \{0, 1, 2, 3, 4, 5\}$. We thus get

$$V^2 = U^3 - 2^{a_1} \cdot 11^{b_1},$$

and we need to determine all the $\{2,11\}$-points (U,V) on the above 36 elliptic curves. Using `SIntegralPoints` subroutine of MAGMA and checking the rank of elliptic curves using `mwrank`, we find that (U,V,a_1,b_1) must be one of the following quadruples:

$$\begin{aligned}&(1,0,0,0),(3,4,0,1),(15,58,0,1),(9/4,5/8,0,1),(345/64,6179/512,0,1),\\&(5,2,0,2),(89/4,835/8,0,2),(5465/16,404003/64,0,2),(11,0,0,3),\\&(443,9324,0,3),(3,5,1,0),(11,33,1,2),(323,5805,1,2),(33/4,143/8,1,2),\\&(2,2,2,0),(5,11,2,0),(785/484,5497/10648,2,0),(5,9,2,1),(82,702,2,4),\\&(2,0,3,0),(33,187,3,2),(22,0,3,3),(25,117,4,2),(33,121,4,3),\\&(465/4,9959/8,4,3),(473,10285,5,3),(2057,93291,5,4).\end{aligned}$$

Identifying the coprime positive integers x and y from the above list and checking the condition $b>0$, one obtains the solutions listed in the statement of the lemma (note that not all of them lead to coprime values for x and y). □

From the above lemma, we can also read the solutions for which $n>3$ is a multiple of 3. Namely, they are obtained from the above list when y is a perfect power. A quick inspection of the list reveals that the only such cases is when $n=3$ and $y=9$ or 25 leading to the solutions $(x,y,a,b,n)=(5,3,6,1,6)$ or $(117,5,4,2,6)$, respectively.

4. The Case $n>3$ Is A Prime

Lemma 5. *If $n>3$ is a prime, then all the solutions to equation* (2) *are* $(x,y,a,b,n)=(1,3,1,2,5)$ *and* $(241,9,3,2,5)$.

Proof. For the beginning of the argument, we only assume that n is not a power of 2. Rewrite equation (2) as

$$x^2+dz^2=y^n,$$

where $d=1,2,11,22$ according to the parities of the exponents a and b. Here, $z=2^\alpha\cdot 11^\beta$ for some nonnegative integers α and β. Thus, our equation becomes

$$(x+i\sqrt{d}z)(x-i\sqrt{d}z)=y^n. \tag{3}$$

Write $\mathbb{K}=\mathbb{Q}[i\sqrt{d}]$. Observe that since $b>0$ and either $a>0$ or x is even, it follows that y is always odd. Thus, the two ideals $(x+iz\sqrt{d})\mathcal{O}_{\mathbb{K}}$ and $(x-iz\sqrt{d})\ \mathcal{O}_{\mathbb{K}}$ are coprime in the ring of integers $\mathcal{O}_{\mathbb{K}}$. Indeed, if $\mathcal{I}$ is some ideal dividing both the above principal ideals, then $\mathcal{I}$ divides both $2x=(x+i\sqrt{d}z)+(x-i\sqrt{dz})$ and y, which are two coprime integers, so $\mathcal{I}=\mathcal{O}_{\mathbb{K}}$. Moreover, the class number of $\mathbb{K}$ is always 1 or 2 and n is coprime to both the class number of $\mathbb{K}$ and to the cardinality of the group of units of $\mathcal{O}_{\mathbb{K}}$, which is 4 or 2 according to whether $d=1$ or $d>1$, respectively, because n is an odd prime. Furthermore, $\{1,i\sqrt{d}\}$ is always an integral basis for $\mathcal{O}_{\mathbb{K}}$ except when $d=11$ in which case an integral basis is $\{1,(1+i\sqrt{11})/2\}$. It thus follows that equation (3) entails that there exist u and v such that

$$x+i\sqrt{d}z=(u+i\sqrt{d}v)^n. \tag{4}$$

Here, either both u and v are integers, or $2u$ and $2v$ are both odd integers, and this last case can occur only when $d=11$. Writing $\lambda=u+i\sqrt{d}v$, conjugating the above relation and eliminating x from the resulting equations, we get that

$$2i\sqrt{d}z=\lambda^n-\overline{\lambda}^n,$$

yielding

$$\frac{2^\alpha \cdot 11^\beta}{v} = \frac{\lambda^n - \overline{\lambda}^n}{\lambda - \overline{\lambda}}. \tag{5}$$

Let us now recall that if λ and $\overline{\lambda}$ are roots of a quadratic equation of the form $x^2 - rx - s = 0$ with nonzero coprime integers r and s and such that $\lambda/\overline{\lambda}$ is not a root of unity, then the sequence $(L_m)_{m\geq 0}$ of general term

$$L_m = \frac{\lambda^m - \overline{\lambda}^m}{\lambda - \overline{\lambda}} \qquad \text{for all } m \geq 0$$

is called a *Lucas sequence*. It can also be defined inductively as $L_0 = 0$, $L_1 = 1$ and $L_{m+2} = L_{m+1} + L_m$. Let us verify that our pair of numbers $(\lambda, \overline{\lambda})$ satisfies the necessary conditions to insure that the right hand side of equation (5) is the nth term L_n of a Lucas sequence. Note that λ and $\overline{\lambda}$ are the roots of

$$x^2 - (\lambda + \overline{\lambda})x + |\lambda|^2 = x^2 - (2u)x + y,$$

and $2u$ and y are coprime integers. Indeed, for if not, since y is odd, it follows that there exists an odd prime q dividing both $2u$ and $y = u^2 + dv^2$. Thus, q divides $4y = (2u)^2 + d(2v)^2$, and since q divides the integer $2u$, it follows that q divides either d or $2v$. In either case, we get that q divides both algebraic integers

$$(2u \pm i\sqrt{d}(2v))^n = 2^n(x \pm i\sqrt{d}z).$$

In particular, q divides the sum of the above two algebraic numbers which is $2^{n+1}x$, and since q is odd, we get that q divides x. This contradicts the fact that x and y are coprime.

Next, we check that $\lambda/\overline{\lambda}$ is not a root of unity. Assume otherwise. Since this number is also in $\mathbb{K}$, it follows that the only possibilities are $\lambda/\overline{\lambda} = \pm 1$, or $\pm i$. The first possibilities give $u = 0$, or $v = 0$, leading to $x = 0$, or $z = 0$, respectively, which are impossible. The second possibility leads to $u = \pm v$, therefore $y = u^2 + v^2 = 2u^2$, or $2y = (2u)^2$. This is again impossible since y is odd and $2u$ is an integer. Hence, indeed the right hand side of equation (5) is the nth term of a Lucas sequence. For any nonzero integer k, let us define $P(k)$ as the largest prime dividing k with convention that $P(\pm 1) = 1$. Equation (5) leads to the conclusion that

$$P(L_n) = P\left(\frac{2^\alpha \cdot 11^\beta}{v}\right) \leq 11. \tag{6}$$

Let us now recall that a prime factor q of L_m is called *primitive* if $q \nmid L_k$ for any $0 < k < m$ and $q \nmid (\lambda - \overline{\lambda})^2 = -4dv^2$. It is known that when q exists, then $q \equiv \pm 1 \pmod{m}$, where the sign coincides with the Legendre symbol $(-d \mid q)$. We now recall that a particular instance of the Primitive Divisor Theorem for Lucas sequences implies that, if $n \geq 5$ is prime, then L_n always has a prime factor except for finitely many *exceptional triples* $(\lambda, \overline{\lambda}, n)$, and all of them appear in Table 1 [13] (see also [1]).

Let us first assume that we are dealing with a number L_n without a primitive divisor. Then a quick look at Table 1 in [13] reveals that this is impossible. Indeed, all exceptional triples have $n = 5, 7$ or 13; of these, there is one example with $n = 5$ such that $\lambda \in \mathbb{Q}[\sqrt{-d}]$ with $d \in \{1, 2, 11, 22\}$, which is $\lambda = \pm(1 \pm i\sqrt{11})/2$. With such a value for λ, we get that $y = |\lambda|^2 = 3$, $d = 11$, therefore the equation is $x^2 + C = 3^5$, where $C = 2^a \cdot 11^b$, with a even and b odd. Since $11^3 > 3^5$, we get that $b = 1$, and next that $a \in \{0, 2, 4\}$, but none of these possibilities yields an integer value for x.

Now let us analyze the possibility when the Lucas number L_n appearing in the right hand side of equation (5) has a primitive divisor. Since $n \geq 5$, it follows that $P(L_n) > 5$, and so $P(L_n) = 11$. Since n is prime and 2 cannot be a primitive divisor of L_n, it follows that 11 is primitive for L_n. Thus, $11 \equiv \pm 1 \pmod{n}$. Since $n \geq 5$ is prime, the only possibility is $n = 5$ and since $11 \equiv 1 \pmod{5}$, we get that $(-d \mid 11) = 1$. Since $d \in \{1, 2, 11, 22\}$, the only possibility is $d = 2$. In particular, u and v are integers. Now since $P(L_n) = 11$ is coprime to $-4dv^2$, we get that $v = \pm 2^{\alpha_1}$ for some $\alpha_1 \leq \alpha$. Reducing equation (5) modulo 2, we get that

$$\pm 2^{\alpha-\alpha_1} 11^{\beta} = \frac{(u + i\sqrt{2}v)^5 - (u - i\sqrt{2}v)^5}{2i\sqrt{2}v} \equiv 5u^4 \pmod{2},$$

and since $y = u^2 + 2v^2$ is odd, we get that u is odd; therefore $\alpha_1 = \alpha$. With $n = 5$ and $v = \pm 2^{\alpha}$, equation (4) becomes

$$\pm 11^{\beta} = 5u^4 - 20u^2v^2 + 4v^4.$$

Note that both when $\alpha = 0$ (so, $v = \pm 1$), and when $\alpha \geq 0$ (so, $4 \mid v^2$), since u is odd it follows that the right hand side of the above equation is congruent to $5 \pmod{8}$. So, $\pm 11^{\beta} \equiv 5 \pmod{8}$, showing that β is odd and the sign on the left hand side is negative. Writing $\beta = 2\beta_0 + 1$, we get that

$$-11V^2 = 5U^4 - 20U^2 + 4,$$

where $(U, V) = (u/v, 11^{\beta_0}/v^2)$ is a $\{2\}$-integer point on the above elliptic curve. With MAGMA, we get that the only such points on the above curve are $(U, V) = (\pm 1, \pm 1)$ and $(\pm 1/2, \pm 1/4)$ leading to $(u, v) = (\pm 1, \pm 1)$ and $(\pm 1, \pm 2)$, respectively. They lead to the desired solutions for $n = 5$, and to the unique solution for $n = 10$. $\square$

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References

[1] M. Abouzaid, *Les nombres de Lucas et Lehmer sans diviseur primitif*, J. Th. Nombres Bordeaux, **18** (2006), 299–313.

[2] F. S. Abu Muriefah, F. Luca, S. Siksek, and S. Tengely, *On the Diophantine equation* $x^2 + C = 2y^n$, Int. J. Number Theory, (to appear).

[3] F. S. Abu Muriefah, F. Luca, and A. Togbe, *On the Diophantine equation* $x^2 + 5^a \cdot 13^b = y^n$, Glasgow Math. J., **50** (2008), 175–181.

[4] F. S. Abu Muriefah, *On the Diophantine equation* $x^2 + 5^{2k} = y^n$, Demonstration Math., **319** (2006), 285–289.

[5] F. S. Abu Muriefah and Y. Bugeaud, *The Diophantine equation* $x^2 + C = y^n$ *a brief overview*, Rev. Colombiana Math., **40** (2006), 31–37.

[6] F. S. Abu Muriefah and S. A. Arif, *The Diophantine equation* $x^2 + q^{2k} = y^n$, Arab. J. Sci. Eng. Sect. A Sci., **26** (2001), 53–62.

[7] S. A. Arif and F. S. Abu Muriefah, *On the Diophantine equation* $x^2 + q^{2k+1} = y^n$, J. Number Theory, **95** (2002), 95–100.

[8] S. A. Arif and F. S. Abu Muriefah, *The Diophantine equation* $x^2 + 5^{2k+1} = y^n$, Indian J. Pure Appl. Math., **30** (1999), 229–231.

[9] S. A. Arif and F. S. Abu Muriefah, *The Diophantine equation* $x^2 + 3^m = y^n$, Int. J. Math. Math. Sci., **21** (1998), 619–620.

[10] S. A. Arif and F. S. Abu Muriefah, *On Diophantine equation* $x^2 + 2^k = y^n$, Int. J. Math. Math. Sci., **20** (1997), 299–304.
[11] M. Bennett and C. Skinner, *Ternary Diophantine equations via Galois representations and modular forms*, Canad. J. Math., **56** (2004), 23–54.
[12] A. Bérczes and I. Pink, *On the Diophantine equation* $x^2 + p^{2k} = y^n$, Arch. Math. (Basel), **91** (2008), 505–517.
[13] Y. F. Bilu, G. Hanrot and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte*, J. Reine Angew. Math., **539** (2001), 75–122.
[14] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations II. The Lebesque-Nagell equation*, Compositio. Math., **142** (2006), 31–62.
[15] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I. The user language*, J. Symbolic Comput., **24** (1997), 235–265.
[16] A. Bérczes, B. Brindza and L. Hajdu, *On the power values of polynomials*, Publ. Math. Debrecen, **53** (1998), 375–381.
[17] J. H. E. Cohn, *The Diophantine equation* $x^2 + C = y^n$, Acta Arith., **65** (1993), 367–381.
[18] J. H. E. Cohn, *The Diophantine equation* $x^2 + 2^k = y^n$, Arch. Math. (Basel), **59** (1992), 341–344.
[19] J. Gebel, E. Hermann, A. Pethő and H. G. Zimmer, *Computing all S-integral points on elliptic curves*, Math. Proc. Camb. Phil. Soc., **127** (1999), 383–402.
[20] E. Goins, F. Luca and A. Togbe, *On the Diophantine equation* $x^2 + 2^\alpha \cdot 5^\beta \cdot 13^\gamma = y^n$, Proceedings of ANTS VIII, A. J. van der Poorten and A. Stein (eds.), Lecture Notes in Computer Science **5011** (2008), 430–442.
[21] K. Győry, I. Pink, and Á. Pintér, *Power values of polynomials and binomial Thue-Mahler equations*, Publ. Math. Debrecen, **65** (2004), 341–362.
[22] E. Landau and A. Ostrowski, *On the Diophantine equation* $ay^2 + by + c = dx^n$, Proc. London Math. Soc., **19** (1920), 276–280.
[23] M. H. Le, *On Cohn's conjecture concerning the Diophantine equation* $x^2+2^m = y^n$, Arch. Math. (Basel), **78** (2002), 26–35.
[24] V. A. Lebesgue, *Sur I'impossibilité en nombres entieres de I'l equation* $x^m = y^2 + 1$, Nouv. Ann. Math., **9** (1850), 178–181.
[25] F. Luca, *On the equation* $x^2 + 2^a \cdot 3^b = y^n$, Int. J. Math. Math. Sci., **29** (2002), 239–244.
[26] F. Luca, *On a Diophantine equation*, Bull. Aus. Math. Soc., **61** (2000), 241–246.
[27] F. Luca and A. Togbe, *On the Diophantine equation* $x^2 + 7^{2k} = y^n$, The Fibonacci Quarterly, **45.4** (2007), 322–326.
[28] F. Luca and A. Togbe, *On the Diophantine equation* $x^2 + 2^a \cdot 5^b = y^n$, Int. J. Number Theory, **4** (2008), 973–979.
[29] M. Mignotte, B. M. M. de Weger, *On the Diophantine equations* $x^2+74 = y^5$ *and* $x^2+86 = y^5$, Glasgow Math. J., **38** (1996), 77–85.
[30] I. Pink, *On the Diophantine equation* $x^2+2^a \cdot 3^b \cdot 5^c \cdot 7^d = y^n$, Publ. Math. Debrecen, **70** (2007), 149–166.
[31] N. Saradha and A. Srinivasan, *Generalized Lebesgue-Ramanujan-Nagell Equations*, in "Diophantine Equations," (Editor N. Saradha) Narosa Publishing House, New Delhi, India, 2008, 207–223.
[32] S. Tengely, *On the Diophantine equation* $x^2 + a^2 = 2y^p$, Indag. Math. (N.S.), **15** (2004), 291–304.

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