TWO MULTIPLE CONVOLUTIONS ON FIBONACCI-LIKE SEQUENCES

WENCHANG CHU AND ROBERTA R. ZHOU

Abstract. Two multiple convolutions on Fibonacci-like sequences are expressed in terms of Stirling numbers of the first kind and the second kind.

1. Introduction and Main Theorems

Let $\{L_n\}_{n\geq 0}$ be the classical Lucas numbers, defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}$$

and the initial values

$$L_0 = 2 \quad \text{and} \quad L_1 = 1.$$

Denote by $\mathbb{N}$ and $\sigma(m,\ell)$, respectively, the set of natural numbers and

$$\sigma(m,\ell) = \Big\{(n_1, n_2, \cdots, n_\ell) \in \mathbb{N}^\ell \,\Big|\, n_1 + n_2 + \cdots + n_\ell = m\Big\}.$$

Recently, Liu [4, Equation 1.4] discovered the following interesting multiple convolution formula

$$\sum_{\sigma(m,\ell)} \prod_{i=1}^{\ell} \frac{L_{n_i}}{n_i} = \ell! \sum_{k=\ell}^{m} (-1)^{k-\ell} \frac{s(k,\ell)}{(2k-m)!(m-k)!} \tag{1}$$

where $s(k,\ell)$ are the Stirling numbers of the first kind. Different multiple convolutions have been treated by Chu and Yan [1]. Motivated by the identity just displayed, this short article will investigate two multiple convolutions on the polynomial sequence $\{G_n\}_{n\geq 0}$ involving four parameters $\{a, b, c, d\}$, which are defined by the recurrence relation

$$G_n(a,b,c,d) = aG_{n-1}(a,b,c,d) + bG_{n-2}(a,b,c,d) \tag{2}$$

with the initial values being given by

$$G_0(a,b,c,d) = c \quad \text{and} \quad G_1(a,b,c,d) = d. \tag{3}$$

For the sake of brevity, we shall further fix two symbols

$$\alpha = \frac{a + \sqrt{a^2+4b}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2+4b}}{2}. \tag{4}$$

Then the first generalized convolution formula reads as the following theorem.

Theorem 1. *Let $\{a, b, c, d\}$ be four complex numbers subject to $a^2 + 4b > 0$ and $ac = 2d$. Then there holds the following convolution formula*

$$\sum_{\sigma(m,\ell)} \prod_{i=1}^{\ell} \frac{G_{n_i}(a,b,c,d)}{n_i} = \Big\{\frac{\alpha d + bc}{\alpha(\alpha-\beta)}\Big\}^{\ell} \ell! \sum_{k=\ell}^{m} (-1)^{k-\ell} \frac{a^{2k-m} b^{m-k}}{(2k-m)!(m-k)!} s(k,\ell).$$

Similarly, we shall establish another generalized multiple convolution theorem.

Theorem 2. *Let $\{a, b, d\}$ be three complex numbers subject to $a^2 + 4b > 0$ and $d \neq 0$. Then there holds the following convolution formula*

$$\sum_{\sigma(m,\ell)} \prod_{i=1}^{\ell} \frac{G_{n_i}(a, b, 0, d)}{n_i!} = d^\ell \ell! \sum_{k=\ell}^{m} \frac{(\alpha - \beta)^{k-\ell}(\beta\ell)^{m-k}}{k!(m-k)!} S(k, \ell)$$

where $S(k, \ell)$ are the Stirling numbers of the second kind.

It is obvious that when $a = b = d = 1$ and $c = 2$, the corresponding $G_n(1, 1, 2, 1)$ become the Lucas numbers L_n. Therefore, Theorem 1 in this case recovers Liu's identity (1). Instead, for $a = b = d = 1$, it is trivial to see that the corresponding $G_n(1, 1, 0, 1)$ are just the classical Fibonacci numbers F_n (see [3, Section 6.6] for example), defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with the initial values

$$F_0 = 0 \quad \text{and} \quad F_1 = 1.$$

In view of Theorem 2, we find the following multiple convolution identity.

Corollary 3. (Multiple convolution on Fibonacci numbers: $\gamma = \frac{1-\sqrt{5}}{2}$.)

$$\sum_{\sigma(m,\ell)} \prod_{i=1}^{\ell} \frac{F_{n_i}}{n_i!} = \ell! \sum_{k=\ell}^{m} 5^{\frac{k-\ell}{2}} \frac{(\gamma\ell)^{m-k}}{k!(m-k)!} S(k, \ell).$$

2. Proof of Theorem 1

For an indeterminate x, define the ordinary generating function by

$$g(x) = \sum_{n=1}^{\infty} G_n(a, b, c, d) x^n.$$

According to (2), it is not difficult to check that $g(x)$ satisfies the equation

$$g(x) = dx + axg(x) + bx^2\{c + g(x)\}.$$

Resolving this equation gives

$$g(x) = \frac{dx + bcx^2}{1 - ax - bx^2}. \tag{5}$$

Further, we can compute another generating function

$$\sum_{n=1}^{\infty} \frac{G_n(a, b, c, d)}{n} x^n = \int_0^x \frac{d + bcx}{1 - ax - bx^2} dx.$$

Taking into account of $a^2 + 4b > 0$, it can be shown that

$$1 - ax - bx^2 = (1 - \alpha x)(1 - \beta x)$$

where α and β are defined by (4). Under the condition $ac = 2d$, there holds the partial fraction decomposition

$$\frac{d + bcx}{1 - ax - bx^2} = \frac{d + bcx}{(1 - \alpha x)(1 - \beta x)} = \frac{d\alpha + bc}{\alpha(\alpha - \beta)} \Big\{ \frac{\alpha}{1 - \alpha x} + \frac{\beta}{1 - \beta x} \Big\}.$$

This leads consequently to

$$\sum_{n=1}^{\infty}\frac{G_n(a,b,c,d)}{n}x^n = \frac{d\alpha+bc}{\alpha(\alpha-\beta)}\int_0^x\left\{\frac{\alpha}{1-\alpha x}+\frac{\beta}{1-\beta x}\right\}dx = \frac{d\alpha+bc}{\alpha(\beta-\alpha)}\log(1-ax-bx^2).$$

For an indeterminate y, define the falling factorials by

$$(y)_0=1 \quad\text{and}\quad (y)_n=y(y-1)\cdots(y-n+1) \quad\text{for}\quad n\in\mathbb{N}.$$

The Stirling numbers of the first kind $s(n,k)$ are defined by (cf. Comtet [2, Section 5.5]

$$(y)_n=\sum_{k=0}^{n}s(n,k)y^k$$

whose exponential generating function reads as

$$\frac{\log^\ell(1+y)}{\ell!}=\sum_{k=\ell}^{\infty}s(k,\ell)\frac{y^k}{k!}.$$

Following Wilf [5], denote by $[x^m]f(x)$ the coefficient of x^m for the formal power series $f(x)$. Then the multiple convolution displayed in Theorem 1 can be expressed as the following coefficient

$$[x^m]\left\{\sum_{n=1}^{\infty}\frac{G_n(a,b,c,d)}{n}x^n\right\}^\ell=\left\{\frac{d\alpha+bc}{\alpha(\beta-\alpha)}\right\}^\ell[x^m]\ \log^\ell(1-ax-bx^2).$$

The last coefficient can further be written in terms of the Stirling numbers of the first kind

$$[x^m]\ \log^\ell(1-ax-bx^2) = \ell!\sum_{k\geq\ell}(-1)^k\frac{s(k,\ell)}{k!}[x^m]\ (ax+bx^2)^k = \ell!\sum_{k=\ell}^{m}(-1)^k\frac{s(k,\ell)}{k!}\binom{k}{m-k}a^{2k-m}b^{m-k}$$

which has been justified by the binomial theorem (cf. [3, Equation 5.12]) as follows

$$[x^m]\ (ax+bx^2)^k=[x^{m-k}]\ (a+bx)^k=\binom{k}{m-k}a^{2k-m}b^{m-k}.$$

This proves the multiple convolution formula stated in Theorem 1.

3. Proof of Theorem 2

When $c=0$ and $d\neq 0$, the generating function $g(x)$ displayed in (5) can similarly be decomposed into partial fractions and then expanded in formal power series

$$\frac{dx}{1-ax-bx^2}=\frac{d}{\alpha-\beta}\left\{\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right\}=\frac{d}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^n-\beta^n)x^n.$$

Extracting the coefficient of x^n leads to the explicit formula

$$G_n(a,b,0,d)=\frac{d}{\alpha-\beta}(\alpha^n-\beta^n).$$

Then we can compute, in turn, the exponential generating function

$$\sum_{n=1}^{\infty}\frac{G_n(a,b,0,d)}{n!}x^n=\frac{d}{\alpha-\beta}\sum_{n=1}^{\infty}(\alpha^n-\beta^n)\frac{x^n}{n!}=\frac{d}{\alpha-\beta}(e^{\alpha x}-e^{\beta x}).$$

In order to evaluate the multiple convolution stated in Theorem 2, recall the Stirling numbers of the second kind $S(n,k)$, which are defined by (cf. Comtet [2, Section 5.2])

$$y^n=\sum_{k=0}^{n}S(n,k)\,(y)_k$$

with the exponential generating function

$$\frac{(e^y-1)^\ell}{\ell!}=\sum_{k\ge\ell}S(k,\ell)\frac{y^k}{k!}.$$

Then we have the following expression

$$\begin{aligned}\sum_{\sigma(m,\ell)}\prod_{i=1}^{\ell}\frac{G_{n_i}(a,b,0,d)}{n_i!} &= [x^m]\Big\{\sum_{n=1}^{\infty}\frac{G_n(a,b,0,d)}{n!}x^n\Big\}^\ell\\ &= [x^m]\Big\{\frac{d}{\alpha-\beta}\Big\}^\ell e^{\beta\ell x}\{e^{(\alpha-\beta)x}-1\}^\ell.\end{aligned}$$

Expanding the last two functions in terms of formal power series and then equating the coefficient of x^m, we obtain

$$\begin{aligned}&\Big\{\frac{d}{\alpha-\beta}\Big\}^\ell[x^m]\Big\{e^{\beta\ell x}\{e^{(\alpha-\beta)x}-1\}^\ell\Big\}\\ =\ &\ell!\Big\{\frac{d}{\alpha-\beta}\Big\}^\ell[x^m]\sum_{i=0}^{\infty}\frac{(\beta\ell x)^i}{i!}\sum_{k=\ell}^{\infty}S(k,\ell)\frac{\{(\alpha-\beta)x\}^k}{k!}\\ =\ &\ell!\Big\{\frac{d}{\alpha-\beta}\Big\}^\ell\sum_{k=\ell}^{m}\frac{(\alpha-\beta)^k(\beta\ell)^{m-k}}{k!(m-k)!}S(k,\ell)\end{aligned}$$

which is equivalent to the right hand side of the equation in Theorem 2.

References

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School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
E-mail address: chu.wenchang@unile.it

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
E-mail address: zhouruiwx1983@yahoo.cn

Current address (W. Chu): Dipartimento Di Matematica, Università del salento, Lecce-Arnesano, PO Box 193, Lecce 73100, Italy