ON GENERALIZED BALANCING SEQUENCES

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Abstract. Let $R_i = R(A, B, R_0, R_1)$ be a second order linear recurrence sequence. In the present paper we prove that any sequence $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0, (A, B) \neq (0, 1)$ is not a balancing sequence.

1. Introduction

In 1999, A. Behera and G. K. Panda [3] defined the notion of balancing numbers. A positive integer n is called a *balancing number* if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+k)$$

for some $k \in \mathbb{N}$. Then k is called the balancer of n. It is easy to see that 6, 35, and 204 are balancing numbers with balancers 2, 14, and 84, respectively. In [3] the authors proved that balancing numbers fulfill the following recurrence relation

$$B_{n+1} = 6B_n - B_{n-1} \quad (n > 1),$$

where $B_0 = 1$ and $B_1 = 6$. In [5], R. Finkelstein studied "The house problem" and introduced the notion of first-power numerical center which coincides with the notion of balancing numbers except for the number 1 which is a first-power numerical center but not a balancing number.

In [8], the authors defined the notion of (k, l)*-power numerical center* or (k, l)*-balancing number*. More precisely let y, k, l be fixed positive integers with $y > 1$. We call the positive integer $x, (x \leq y)$, a (k, l)*-power numerical center* or (k, l)*-balancing number* for y if

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

In [5], R. Finkelstein proved that there are no second-power numerical centers (in this case $k = l = 2$). Later on R. Steiner [13], proved that there are no third-power numerical centers (in this case $k = l = 3$). (Here we mention that R. Finkelstein and R. Steiner is the same person.) In the case $k = 4$ and $k = 5$ he conjectured a negative answer. Later on P. Ingram in [6] using the explicit lower bounds on linear forms in elliptic logarithms, proved that there are no nontrivial fifth-power numerical centers. In the same paper he proved that there are only finitely many nth power numerical centers.

K. Liptai, F. Luca, Á. Pintér, and L. Szalay [8] obtained certain effective and ineffective finiteness theorems for (k, l) numerical centers. Their results are based on Baker's theory and a result of Cs. Rakaczki [11], respectively. Furthermore, they proved that there exists no (k, l) numerical center with $l > k$.

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In [7], K. Liptai searched for those balancing numbers which are Fibonacci numbers, too. Using the results of A. Baker and G. Wüstholz [2] he proved that there are no Fibonacci balancing numbers. Using another method L. Szalay [14] proved that there are no Lucas balancing numbers.

Later G. K. Panda and P. K. Ray [9] slightly modified the definition of a balancing number and introduced the notion of a cobalancing number. A positive integer n is called a *cobalancing number* if

$$1 + 2 + \cdots + (n-1) + n = (n+1) + (n+2) + \cdots + (n+K)$$

for some $K \in \mathbb{N}$. In this case K is called the cobalancer of n.

They also proved that the cobalancing numbers fulfill the following recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad (n > 1),$$

where $b_0 = 1$ and $b_1 = 6$. Moreover they found that every balancer is a cobalancing number and every cobalancer is a balancing number.

In [10], G. K. Panda gave another possible generalization of balancing numbers. Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of real numbers. We call an element a_n of this sequence a *sequence-balancing* number if

$$a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + a_{n+2} + \cdots + a_{n+k}$$

for some $k \in \mathbb{N}$. Similarly, one can define the notion of *sequence cobalancing numbers.* In [10] it was proved that there does not exist any sequence balancing number in the Fibonacci sequence. The sequence $R = \{R_i\}_{i=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence sequence if the recurrence relation

$$R_i = AR_{i-1} + BR_{i-2} \ (i \geq 2)$$

holds, where $A, B \neq 0, R_0, R_1$ are fixed rational integers and $|R_0| + |R_1| > 0$. The polynomial $f(x) = x^2 - Ax - B$ is called the companion polynomial of the sequence $R = R(A, B, R_0, R_1)$. Let $D = A^2 + 4B$ be the discriminant of f. The roots of the companion polynomial will be denoted by α and β. Using this notation if $D \neq 0$, as it is well-known, we may write

$$R_i = \frac{a\alpha^i - b\beta^i}{\alpha - \beta}$$

for $i \geq 2$, where $a = R_1 - R_0\beta$ and $b = R_1 - R_0\alpha$.

As a generalization of the notion of a balancing number, we will call a binary recurrence $R_i = R(A, B, R_0, R_1)$ a *balancing sequence* if

$$R_1 + R_2 + \cdots + R_{n-1} = R_{n+1} + R_{n+2} + \cdots + R_{n+k} \tag{1}$$

holds for some $k \geq 1$ and $n \geq 2$.

In the present paper we prove that any sequence $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0$, $(A, B) \neq (0, 1)$ is not a balancing sequence.

2. Results

Theorem 1. *There is no balancing sequence of the form $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0$ except for $(A, B) = (0, 1)$ in which case (1) has infinitely many solutions $(n, k) = (n, n-1)$ and $(n, k) = (n, n)$ for $n \geq 2$.*

As a consequence of Theorem 1 above, we consider the question of Lucas-sequences. As it is well-known, a sequence

$$R_i = R(A, B, 0, 1) = \frac{\alpha^i - \beta^i}{\alpha - \beta}$$

is called a Lucas-sequence, if $\frac{\alpha}{\beta}$ is not a root of unity and $\gcd(A, B) = 1$.

Corollary 1. *Let $R_i = R(A, B, 0, 1)$ be a Lucas-sequence with $A^2 + 4B > 0$. Then R_i is not a balancing sequence.*

3. Auxiliary Results

Lemma 1. *Let $n \geq 2$ and $k \geq 1$ be integers and consider the function $F : \mathbb{R} \setminus \{1\} \to \mathbb{R}$,*

$$F(x) = \frac{x^{n+k+1} - x^{n+1} - x^n + x}{x - 1}.$$

Then F is strictly increasing on the interval $(-\infty, -1]$ if $n + k$ is odd and F is strictly decreasing on the interval $(-\infty, -1]$ if $n + k$ is even.

Proof. The derivative F' of F is

$$F'(x) = \frac{(n + k)x^{n+k+1} - (n + k + 1)x^{n+k} - nx^{n+1} + 2x^n + nx^{n-1} - 1}{(x - 1)^2}.$$

We may suppose that $x \leq -1$. Hence we have $x = -|x|$. Therefore $F'(x)$ can be rewritten in the form

$$F'(x) = \frac{|x|^{n+k}g(x) - |x|^{n+1}h(x)}{(x - 1)^2},$$

where

$$g(x) = (-1)^{n+k}(-n-k-1-(n+k)|x|) \text{ and } h(x) = n(-1)^{n+1} - \frac{2(-1)^n}{|x|} - \frac{n(-1)^{n-1}}{|x|^2} + \frac{1}{|x|^{n+1}}. \tag{2}$$

Now, if $n + k$ is odd, then since $|x| \geq 1$ and since $n + 1$ and $n - 1$ have the same parity, by (2) one gets

$$g(x) \geq 2n + 2k + 1 \text{ and } h(x) < n + 2 + 1 = n + 3.$$

Hence,

$$F'(x) > \frac{(n + 2k - 2)|x|^{n+1}}{(x - 1)^2},$$

so for $k \geq 1$ and $n \geq 2$ this leads to $F'(x) > 0$ for $x \leq -1$ and the lemma follows.

Finally, if $n + k$ is even and $|x| \geq 1$ we have

$$g(x) \leq -2n - 2k - 1 \text{ and } h(x) > -n - 2 + 1 = -n - 1,$$

so for $k \geq 1$ we get

$$F'(x) < \frac{(-n - 2k)|x|^{n+1}}{(x - 1)^2}.$$

Since $n \geq 2$ and $k \geq 1$ one observes that $F'(x) < 0$ holds for $x \leq -1$, so the lemma follows. □

4. Proof of Theorem 1

Consider the sequence $R_i = R(A, B, 0, R_1)$ with $R_1 \neq 0$, companion polynomial $f(x) = x^2 - Ax - B$, and positive discriminant $D = A^2 + 4B > 0$. Since $R(A, B, 0, R_1) = R_1 \cdot R(A, B, 0, 1)$ one can observe that $R(A, B, 0, R_1)$ is a balancing sequence (i.e (1) holds) if and only if $R(A, B, 0, 1)$ is a balancing sequence. Thus we may assume that $R_1 = 1$ that is, in what follows we may deal without loss of generality with the sequence $R_i = R(A, B, 0, 1)$.

We distinguish several subcases according to $A = 0$ or to the signs of A and B, respectively.

Case 1: $A = 0$.

Since $0 < D = A^2 + 4B$ it follows that $B > 0$. The roots of the companion polynomial $f(x) = x^2 - B$ are $\alpha = \sqrt{B}$ and $\beta = -\alpha = -\sqrt{B}$. Thus we have the sequence

$$R_i = \frac{\sqrt{B}^i - (-\sqrt{B})^i}{2\sqrt{B}}, \ i \geq 0.$$

Now, if $B = 1$ then R_i is of the form

$$R_i = \frac{1^i - (-1)^i}{2}, \ i \geq 0$$

which is obviously a balancing sequence. Further, the resulting equation (1) in this case has infinitely many solutions (n, k), namely $(n, k) = (n, n - 1)$ and (n, n) for $n \geq 2$.

If $B > 1$ then for $i \geq 0$ we have

$$R_i = \begin{cases} 0, & \text{if } i \text{ is even,} \\ B^{\frac{i-1}{2}}, & \text{if } i \text{ is odd.} \end{cases}$$

Suppose that (1) holds with $n \geq 2$ odd. Since in this case $R_{n-1} = R_{n+1} = 0$ and the left hand side of (1) is $\frac{B^{\frac{n-1}{2}}-1}{B-1}$, we may obviously assume that $k \geq 2$. Now, for the right hand side of (1) we have

$$R_{n+1} + R_{n+2} + \cdots + R_{n+k} \geq R_{n+2} = B^{\frac{n+1}{2}}.$$

But this leads to a contradiction in view of (1), $B > 1$, $n \geq 2$ and

$$\frac{B^{\frac{n-1}{2}} - 1}{B - 1} < B^{\frac{n+1}{2}}.$$

Finally, if equation (1) holds with $n \geq 2$ even then $R_{n-1} = B^{\frac{n-2}{2}}$ and $R_{n+1} = B^{\frac{n}{2}}$. Hence the left hand side of (1) is $\frac{B^{\frac{n}{2}}-1}{B-1}$ while for the right hand side we have the lower bound $B^{\frac{n}{2}}$. This is impossible by (1), $B > 1$, $n \geq 2$ and

$$\frac{B^{\frac{n}{2}} - 1}{B - 1} < B^{\frac{n}{2}}.$$

Hence, in this case there is no balancing sequence apart from $B = 1$.

Case 2: $A > 0$.

Let α and β be the roots of the companion polynomial $f(x) = x^2 - Ax - B$. One observes that f has a dominant root which we will denote by α. (Note that α is a dominant root of f if $|\alpha| > |\beta|$). In this case we have

$$\alpha = \frac{A + \sqrt{A^2 + 4B}}{2}, \ \beta = \frac{A - \sqrt{A^2 + 4B}}{2}.$$

Since $A \geq 1$ and $D = A^2 + 4B > 0$ we obviously have $\alpha > 1$. Further, since $R_i = \frac{\alpha^i - \beta^i}{\alpha - \beta}$ $(i \geq 0)$ and $|\beta| < \alpha$ we get that $R_i > 0$ for $i \geq 1$. Suppose that (1) holds for some $n \geq 2$ and $k \geq 1$. We derive an upper bound for the left hand side of (1). Since

$$R_i = \frac{\alpha^i - \beta^i}{\alpha - \beta} < \frac{2\alpha^i}{\alpha - \beta},$$

we get

$$R_1 + R_2 + \cdots + R_{n-1} < \frac{2}{\alpha - \beta} \sum_{i=1}^{n-1} \alpha^i = \left(\frac{2\alpha}{\alpha - \beta} \right) \left(\frac{\alpha^{n-1} - 1}{\alpha - 1} \right). \tag{3}$$

Further, since $R_i > 0$ for all $i \geq 1$ we get for the right hand side of (1) the lower bound

$$R_{n+1} + R_{n+2} + \cdots + R_{n+k} \geq \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Suppose first that $\beta^{n+1} < 0$. Then

$$R_{n+1} + R_{n+2} + \cdots + R_{n+k} > \frac{\alpha^{n+1}}{\alpha - \beta}. \tag{4}$$

Further, we see that $\beta^{n+1} < 0$ holds if and only if $\beta < 0$ (and $n + 1$ odd). Hence, we may assume that $B > 0$. Now, by (1), (3), and (4) we obtain

$$\frac{\alpha^{n+1}}{\alpha - \beta} < \left(\frac{2\alpha}{\alpha - \beta} \right) \left(\frac{\alpha^{n-1} - 1}{\alpha - 1} \right),$$

which leads to

$$\alpha^2 - \alpha - 2 < -\frac{2}{\alpha^{n-1}}. \tag{5}$$

Thus (5) implies that $\alpha = \frac{A + \sqrt{A^2 + 4B}}{2} < 2$ and since $A > 0$ and $B > 0$ this can occur only if $A = B = 1$. In this case the resulting sequence is the Fibonacci sequence and for it

$$R_1 + \cdots + R_{n-1} = F_1 + \cdots + F_n = F_{n+1} - 1 < F_{n+1} = R_{n+1}. \tag{6}$$

Thus (6) shows that there is no balancing sequence if $\beta^{n+1} < 0$.

Suppose now that $\beta^{n+1} > 0$ and assume that (1) holds for some $n \geq 2$ and $k \geq 1$. In this case the upper bound (3) for the left-hand side of (1) remains valid. Since $\alpha > |\beta|$ for R_{n+1} we get the following lower bound

$$R_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{\alpha^{n+1} - |\beta|^{n+1}}{\alpha - \beta} = \frac{(\alpha - |\beta|)(\alpha^n + \cdots + |\beta|^n)}{\alpha - \beta} > \frac{\Delta \alpha^n}{\alpha - \beta}, \tag{7}$$

where $\Delta = \alpha - |\beta| = \sqrt{A^2 + 4B}$. Hence, using (1), (3), and (7) we get

$$\frac{\Delta \alpha^n}{\alpha - \beta} < \left(\frac{2\alpha}{\alpha - \beta} \right) \left(\frac{\alpha^{n-1} - 1}{\alpha - 1} \right)$$

which leads to

$$\Delta\alpha^{n+1} - (\Delta+2)\alpha^n < -2\alpha. \tag{8}$$

But (8) is a contradiction if $\alpha \geq \frac{\Delta+2}{\Delta} = 1 + \frac{2}{\Delta}$. Finally, if $\alpha < 1 + \frac{2}{\Delta}$ then since $\Delta \geq 1$ we get that $\alpha < 3$. Thus, those values of the pair (A, B) for which $\alpha = \frac{A+\sqrt{A^2+4B}}{2} < 3$ and $A^2 + 4B > 0$ are the following

$$(A, B) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (3, -2), (3, -1)\}.$$

Now, if $(A, B) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (3, -1)\}$ we see that $\Delta \geq \sqrt{5}$ and hence

$$\alpha < 1 + \frac{2}{\sqrt{5}}, \tag{9}$$

which implies that the only value of $\alpha = \frac{A+\sqrt{A^2+4B}}{2}$ for which (9) holds is $A = B = 1$, i.e. $\alpha = \frac{1+\sqrt{5}}{2}$. But in this case the resulting sequence is again the Fibonacci sequence for which we have already checked that (1) cannot hold.

Finally, if $(A, B) = (3, -2)$ the resulting sequence is $R_i = 2^i - 1$ for $i \geq 0$. Assume that (1) holds for this sequence. One can easily see that the left hand side of (1) in this case is $2^n - n - 1$. Further, for the right hand side of (1) we have the lower bound $R_{n+1} = 2^{n+1} - 1$. But for $n \geq 2$

$$2^{n+1} - 1 > 2^n - n - 1$$

which shows that the sequence $R_i = 2^i - 1$ cannot be a balancing sequence. So there is no balancing sequence with $A > 0$ and $D = A^2 + 4B > 0$.

Case 3: $A < 0$ and $B < 0$.

We work as in the previous case. Let α denote the dominant root of the companion polynomial $f(x) = x^2 - Ax - B$. Since $A < 0$ and $A^2 + 4B > 0$ we have

$$\alpha = \frac{A - \sqrt{A^2+4B}}{2} \text{ and } \beta = \frac{A + \sqrt{A^2+4B}}{2}.$$

Since $-B = \alpha\beta$ we see that $\beta < 0$. Now, if R_i is a balancing sequence then by (1) and $R_i = \frac{\alpha^i - \beta^i}{\alpha - \beta}$ we get

$$\frac{\alpha^{n+k+1} - \alpha^{n+1} - \alpha^n + \alpha}{\alpha - 1} = \frac{\beta^{n+k+1} - \beta^{n+1} - \beta^n + \beta}{\beta - 1}, \tag{10}$$

where $n \geq 2$ and $k \geq 1$. Thus from (10) we have

$$F(\alpha) = F(\beta), \tag{11}$$

where F is the function defined in Section 3. Now, if $\beta \leq -1$ then by $\alpha < \beta$ we get by Lemma 1 that $F(\alpha) < F(\beta)$ if $n + k$ is odd and $F(\alpha) > F(\beta)$ if $n + k$ is even. But this contradicts (11). Hence, we may assume that $-1 < \beta < 0$ and we may suppose without loss of generality that $\alpha \leq \alpha_0 = \frac{-3-\sqrt{5}}{2}$. By $k \geq 1$, $n \geq 2$, and $|\alpha| \geq |\alpha_0|$ we have

$$|1 - 1/\alpha^k - 1/\alpha^{k+1} + 1/\alpha^{n+k}| > 0.4. \tag{12}$$

Since $-1 < \beta < 0$ we have $|\beta| < 1$ and $|\beta - 1| > 1$. Hence we get by (10), (11), and (12)

$$\frac{0.4|\alpha|^{n+k+1}}{|\alpha| + 1} < |F(\alpha)| = |F(\beta)| < \frac{4}{|\beta - 1|} < 4. \tag{13}$$

But (13) is a contradiction in view of $n \geq 2$, $k \geq 1$, and $|\alpha| \geq |\alpha_0| = \frac{3+\sqrt{5}}{2}$. Hence there are no balancing sequences with $A < 0, B < 0$ and $A^2 + 4B > 0$.

Case 4: $A < 0$ and $B > 0$.

Let us now consider the sequence $R_i = R(A, B, 0, 1)$ with $A < 0$ and $B > 0$. We also consider the corresponding sequence $Q_i := R(|A|, B, 0, 1)$. We clearly have $R_i = (-1)^{i-1} Q_i$ ($i \geq 1$) and thus $|R_i| = |Q_i| = Q_i$. Further, by induction on i it is easily seen that

$$Q_1 + Q_2 + \cdots + Q_{i-1} < Q_{i+1} \quad \text{for} \quad i = 2, 3, \ldots. \tag{14}$$

First we suppose $A \leq -2$. Now the absolute value of the left hand side of (1) is

$$|R_1 + \cdots + R_{n-1}| \leq Q_1 + \cdots + Q_{n-1}. \tag{15}$$

Further, by $Q_{i+1} = |A|Q_i + BQ_{i-1} \geq 2Q_i$ we have $Q_{i+1} - Q_i \geq Q_i$ for $i \in \mathbb{N}$ and the absolute value of the right hand side of (1) is

$$|R_{n+1} + \cdots + R_{n+k}| = |Q_{n+1} - Q_{n+2} + \cdots + (-1)^{k-1} Q_{n+k}|$$

and this is one of the following:

$$Q_{n+1}, \quad Q_{n+2} - Q_{n+1} \geq Q_{n+1}, \quad Q_{n+3} - Q_{n+2} + Q_{n+1} \geq Q_{n+1}, \ \ldots.$$

This together with (14) and (15) concludes the proof of Case 4 if $A \leq -2$.

Finally, if $A = -1$ then $Q_{i+1} - Q_i = BQ_{i-1}$ for $i \in \mathbb{N}$. Now the absolute value of the left hand side of (1) is

$$\begin{aligned} |R_1 + \cdots + R_{n-1}| &= |Q_{n-1} - Q_{n-2} + Q_{n-3} - Q_{n-4} + \cdots| \\ &\leq (Q_{n-1} - Q_{n-2}) + (Q_{n-3} - Q_{n-4}) + \cdots \\ &\leq B\,(Q_{n-3} + Q_{n-2} + \cdots + Q_1) < BQ_{n-1}. \end{aligned} \tag{16}$$

On the other hand, the right hand side of (1) is again one of the following:

$$Q_{n+1} > BQ_{n-1}, \quad Q_{n+2} - Q_{n+1} > BQ_{n-1}, \quad Q_{n+3} - Q_{n+2} + Q_{n+1} > BQ_{n-1}, \ \ldots.$$

This together with (14) and (16) concludes the proof of our Theorem 1.

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