DIGIT PROPORTIONS IN ZECKENDORF REPRESENTATIONS

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Abstract. In this paper we study various structural properties of an infinite matrix M of 0's and 1's, the rows of which correspond to the Zeckendorf representations of the non-negative integers. In particular, we calculate the proportion of 1's appearing amongst the digits in each of the columns of M and go on to obtain some associated asymptotic results.

1. Introduction

The most well-known method for representing non-negative integers using just the digits 0 and 1 is of course by way of the bicimal system. With the subscript 2 denoting the fact that the numbers are given in binary, we have

$$\begin{aligned} 0_2 &= 0 \\ 1_2 &= 1 \\ 10_2 &= 2 \\ 11_2 &= 3 \\ 100_2 &= 4 \\ 101_2 &= 5 \\ 110_2 &= 6 \\ 111_2 &= 7 \\ 1000_2 &= 8 \end{aligned}$$

and so on. It is clear that when $n \in \mathbb{N}$ is large, the probability that a number chosen at random from $\{0, 1, 2, \ldots, n\}$ will end in 1 when written in binary is approximately one-half. In fact, by taking n sufficiently large, the probability that a randomly-chosen number from $\{0, 1, 2, \ldots, n\}$ will have a 1 in some specified position can be made arbitrarily close to one-half.

The above result is certainly rather straightforward, but there are other ways of representing integers using just 0's and 1's for which the corresponding situation is not quite so obvious. One such system is the Zeckendorf representation. In this paper we study the structural properties of an infinite matrix for which the rows correspond to the Zeckendorf representations of the non-negative integers. These properties are then used to derive results concerning the proportion of 1's appearing amongst the digits in such representations.

2. Zeckendorf Representations of Integers

Zeckendorf's theorem is a reasonably well-known result concerning the possibility of writing positive integers as a sum of distinct Fibonacci numbers. Both the theorem and the representation are named after the Belgian medical doctor and amateur mathematician Edouard

Zeckendorf (1901–1983). Although he did not publish anything in this regard until 1972 [8], it would appear that he had first obtained a proof as early as 1939. A short, nonetheless interesting, biography of Zeckendorf is to be found in [3].

The theorem states that every $n \in \mathbb{N}$ can be represented in a unique way as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. Somewhat more formally and precisely, for any $n \in \mathbb{N}$ there exists an increasing sequence of positive integers of length $k \in \mathbb{N}$, $(a_1, a_2, \ldots, a_k)$ say, such that $a_1 \geq 2$, $a_i \geq a_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and

$$n = \sum_{i=1}^{k} F_{a_i}.$$

A proof is given in both [1] and [7].

For example, consider the following two ways of writing 43 as a sum of distinct Fibonacci numbers:

$$\begin{aligned} 43 &= 1 + 8 + 34 \\ &= F_2 + F_6 + F_9 \end{aligned}$$

and

$$\begin{aligned} 43 &= 1 + 3 + 5 + 34 \\ &= F_2 + F_4 + F_5 + F_9. \end{aligned}$$

The consecutive Fibonacci numbers appearing in the second sum above preclude this from being a Zeckendorf representation. It is in fact clear that the first sum is a Zeckendorf representation of 43. Furthermore, by Zeckendorf's theorem, this is the only way (other than the order of the terms) that 43 can be expressed in this way. We may thus write

$$43 = 10010001_Z,$$

with the subscript denoting the fact that this is a Zeckendorf representation.

Note that no Zeckendorf representation requires the use of F_1. Clearly, if a Zeckendorf representation of n contains F_2 then on replacing F_2 with F_1 we would still have a representation for n that does not include any consecutive Fibonacci numbers. This would, however, violate the uniqueness of these representations. It is thus necessary to stipulate that F_1 does not appear in any Zeckendorf representation. (Of course, an alternative would be to allow F_1 to be used rather than F_2, but the convention is for the F_2 column to denote the 'units'.) We will have cause to use the following result:

(Z1) The Zeckendorf representation of any $n \in \mathbb{N}$ contains the largest Fibonacci number not exceeding n.

3. The Golden String

Definition 3.1. *Let A and B be finite strings of symbols. We use $A : B$ to denote the concatenation of A and B.*

The *golden string*, $S_\infty = abaababaabaab\ldots$, may be obtained recursively as follows. We start with the strings $S_1 = a$ and $S_2 = ab$. In order to obtain S_3 we concatenate S_2 and S_1

as follows:

$$\begin{aligned} S_3 &= S_2 : S_1 \\ &= aba. \end{aligned}$$

Next,

$$\begin{aligned} S_4 &= S_3 : S_2 \\ &= abaab, \end{aligned}$$

and so on. In general $S_k = S_{k-1} : S_{k-2}$ for $k \geq 2$, and it is clear that S_k contains F_{k+1} letters (F_k a's and F_{k-1} b's).

It is more usual to use 1's and 0's than a's and b's as elements of the golden string; see [4], for example. This convention is not adopted here however, as it would create the potential for notational confusion later in the paper. We will need the following simple result associated with the golden string [5]:

(G1) The number of a's in the first n elements of S_∞ is given by

$$\left\lfloor \frac{n+1}{\phi} \right\rfloor,$$

where

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the golden ratio.

4. The Zeckendorf Matrix

From Section 2 we know that for any non-negative integer n there exists a unique sequence (c_k) for which $c_k \in \{0,1\}$ and $c_k c_{k+1} = 0$, $k = 1, 2, 3, \ldots$, such that

$$n = \sum_{k=2}^{\infty} c_k F_k.$$

We construct from this the infinite *Zeckendorf matrix* M whose entry $m(i,j)$ gives the coefficient c_j of F_j in the Zeckendorf representation of n. So, for example, $m(43,6) = 1$ and $m(43,7) = 0$. Note that, for the sake of convenience, the rows of M are numbered from 0 onwards, while the columns are numbered from 2 onwards. It ought also to be borne in mind that the convention of numbering columns of matrices from left to right means that the nth row of M corresponds to a 'reflection' of the Zeckendorf representation of n. As will be shown in due course, the structure of each column of M is closely related to the golden string.

Definition 4.1. *For $a \leq b$, let $D_k(a,b)$ denote the string consisting of the ath to the bth digits inclusive in the kth column of M. We use $D_k(0,\infty)$ to represent the string comprising the whole of the kth column of M.*

Lemma 4.2. *For $p \geq 1$ and $k \geq 2$ we have*

$$D_k\left(0, F_{k+p} - 1\right) = D_k\left(F_{k+p+1}, F_{k+p+2} - 1\right).$$

Proof. Consider any $n \in \{0, 1, \ldots, F_{k+p} - 1\}$. By Zeckendorf's theorem there exists an increasing sequence of positive integers of length $r \in \mathbb{N}$, $(a_1, a_2, \ldots, a_r)$ say, such that $a_1 \geq 2$, $a_i \geq a_{i-1} + 2$ for $i = 2, 3, \ldots, r$, and

$$n = \sum_{i=1}^{r} F_{a_i}.$$

Furthermore, since $a_r \leq k + p - 1$, the expression

$$\sum_{i=1}^{r} F_{a_i} + F_{k+p+1}$$

gives the Zeckendorf representation for $n + F_{k+p+1}$. Therefore,

$$m(n, k) = m\left(n + F_{k+p+1}, k\right),$$

which proves the lemma. □

Definition 4.3. *Let $A_k = F(0, k) : F(1, k-1) : F(0, k)$ and $B_k = F(0, k) : F(1, k-1)$, where $F(0, k)$ and $F(1, k)$ denote strings of F_k 0's and 1's, respectively. Thus, for example, $A_4 = 00011000$ and $B_5 = 00000111$.*

Definition 4.4. *We use $S_i(A_k, B_k)$ to denote the string of 0's and 1's obtained by replacing each a and b in S_i with A_k and B_k, respectively. In a similar manner, $S_\infty(A_k, B_k)$ may be obtained from S_∞. For example,*

$$S_3(A_4, B_4) = 000110000001100011000.$$

Theorem 4.5.

$$S_\infty(A_k, B_k) = D_k(0, \infty)$$

for all $r \in \mathbb{N}$.

Proof. Let us consider the kth column of M for some $k \geq 2$, remembering that this gives us, for each $n \in \mathbb{N}_0$, the coefficient of F_k in the Zeckendorf representation of n. We proceed by induction on r, showing that

$$S_r(A_k, B_k) = D_k\left(0, F_{k+r+1} - 1\right)$$

for each $r \in \mathbb{N}$.

First, it is clear that $D_k(0, F_k - 1)$ is a string of 0's. Second, from (Z1) it follows that $D_k(F_k, F_{k+1} - 1)$ consists entirely of 1's. Third, using (Z1) once more, we know that $D_{k+1}(F_{k+1}, F_{k+2} - 1)$ consists entirely of 1's, and thus $D_k(F_{k+1}, F_{k+2} - 1)$ must consist entirely of 0's. It is therefore the case that

$$S_1(A_k, B_k) = D_k\left(0, F_{k+2} - 1\right).$$

Following similar arguments, it can be shown that $D_k(F_{k+2}, F_{k+3} - 1) = B_k$, and hence that

$$S_2(A_k, B_k) = D_k\left(0, F_{k+3} - 1\right).$$

Let us now assume that, for some $r \in \mathbb{N}$, it is true that

$$S_i(A_k, B_k) = D_k\left(0, F_{k+i+1} - 1\right)$$

for $i = 1, 2, \ldots, r$. Then

$$\begin{aligned} S_{r+1}(A_k, B_k) &= S_r(A_k, B_k) : S_{r-1}(A_k, B_k) \\ &= D_k(0, F_{k+r+1} - 1) : D_k(0, F_{k+r} - 1) \\ &= D_k(0, F_{k+r+1} - 1) : D_k(F_{k+r+1}, F_{k+r+2} - 1) \\ &= D_k\left(0, F_{k+(r+1)+1} - 1\right), \end{aligned}$$

where we have used Lemma 4.2 along with Definition 4.4. The result has thus been proved. □

5. Formulas for the Proportions of 1's

Theorem 5.1. *Let, for $k \geq 2$, p_k denote the proportion of 1's appearing amongst the digits in column k of M. Then*

$$p_k = \frac{F_{k-1}}{\phi^k}.$$

Proof. From Definition 4.3 it can be seen that the numbers of digits in A_k and B_k are given by

$$F_k + F_{k-1} + F_k = F_{k+2} \quad \text{and} \quad F_k + F_{k-1} = F_{k+1},$$

respectively.

We know from Theorem 4.5 that $D_k(0, \infty)$ consists essentially of a series of blocks, with each block corresponding either to A_k or B_k. From (G1) it is also true that, of the first q of these blocks, $\left\lfloor \frac{q+1}{\phi} \right\rfloor$ correspond to A_k's. The number of digits $h_k(q)$ in the first q blocks is thus given by

$$\begin{aligned} h_k(q) &= \left\lfloor \frac{q+1}{\phi} \right\rfloor F_{k+2} + \left(q - \left\lfloor \frac{q+1}{\phi} \right\rfloor\right) F_{k+1} \\ &= qF_{k+1} + \left\lfloor \frac{q+1}{\phi} \right\rfloor (F_{k+2} - F_{k+1}) \\ &= qF_{k+1} + \left\lfloor \frac{q+1}{\phi} \right\rfloor F_k. \end{aligned}$$

Both A_k and B_k contain F_{k-1} 1's. Therefore, in the first $h_k(q)$ digits of $D_k(0, \infty)$ there are qF_{k-1} 1's. The proportion of 1's in column k to this point is thus

$$\frac{qF_{k-1}}{qF_{k+1} + \left\lfloor \frac{q+1}{\phi} \right\rfloor F_k}.$$

From this it follows that

$$\begin{aligned} p_k &= \lim_{q \to \infty} \frac{qF_{k-1}}{qF_{k+1} + \left\lfloor \frac{q+1}{\phi} \right\rfloor F_k} \\ &= \frac{F_{k-1}}{F_{k+1} + \left(\frac{F_k}{\phi}\right)} \\ &= \frac{F_{k-1}}{\phi^k}, \end{aligned}$$

where we have used the result $\phi F_{k+1} + F_k = \phi^{k+1}$. □

Corollary 5.2. *The proportion of 1's in columns 2 to k inclusive, $\overline{p}_k$ say, is given by*

$$\overline{p}_k = \frac{1}{k-1} \sum_{j=2}^{k} \frac{F_{j-1}}{\phi^j}.$$

Proof. This result follows immediately from Theorem 5.1. □

Corollary 5.3. *For large k, p_k is approximately equal to $\frac{5-\sqrt{5}}{10}$.*

Proof.

$$\begin{aligned}\lim_{k\to\infty} p_k &= \lim_{k\to\infty} \frac{F_{k-1}}{\phi^k} \\ &= \frac{1}{\phi\sqrt{5}} \\ &= \frac{5-\sqrt{5}}{10}.\end{aligned}$$

□

Finally, it is worth noting the following:

(1) One interpretation of Corollary 5.3 is that for any $\epsilon > 0$ there exists a pair (n, k) of positive integers such that when a number is chosen at random from the set $\{0, 1, 2, \ldots, n\}$ the expected proportion of 1's in the rightmost k digits of the Zeckendorf representation of this number is within ϵ of $\frac{5-\sqrt{5}}{10}$.

(2) Since $\sum_{j=2}^{\infty} \frac{1}{\phi^j} = 1$, the expression $\sum_{j=2}^{k} \frac{F_{j-1}}{\phi^j}$ may, for large k, be regarded as an exponentially-weighted average of the first $k-1$ Fibonacci numbers.

(3) The proportions $\overline{p}_k$ and p_k exhibit different behavior in that the former approaches $\frac{5-\sqrt{5}}{10}$ monotonically as k increases, while the latter tends to this limit in an oscillatory manner.

(4) The following Fibonacci triangle (see A058071 in [6]) arises naturally from M:

$$\begin{array}{c} 1 \\ 1\quad 1 \\ 2\quad 1\quad 2 \\ 3\quad 2\quad 2\quad 3 \\ 5\quad 3\quad 4\quad 3\quad 5 \\ 8\quad 5\quad 6\quad 6\quad 5\quad 8 \\ 13\quad 8\quad 10\quad 9\quad 10\quad 8\quad 13 \\ 21\quad 13\quad 16\quad 15\quad 15\quad 16\quad 13\quad 21 \\ 34\quad 21\quad 26\quad 24\quad 25\quad 24\quad 26\quad 21\quad 34 \\ 55\quad 34\quad 42\quad 39\quad 40\quad 40\quad 39\quad 42\quad 34\quad 55 \\ 89\quad 55\quad 68\quad 63\quad 65\quad 64\quad 65\quad 63\quad 68\quad 55\quad 89 \\ \vdots \end{array}$$

The kth entry in the rth row of this triangle is equal to $F_k F_{r-k+1}$, which is also equal to the number of 1's in $D_k(0, F_{r+2})$. The rth row-sum of the above triangle is easily shown to be

$$\sum_{k=1}^{r} F_k F_{r-k+1} = \frac{1}{5}\left(rF_{r+1} + 2(r+1)F_r\right)$$

(see [2] or [5], for example), which coincides with the total number of 1's contained in the first F_{r+2} rows of M. This is the sequence of Fibonacci numbers convolved with themselves, and appears in [6] as A001629.

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