ON FIBONACCI KNOTS

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ABSTRACT. We show that the Conway polynomials of Fibonacci links are Fibonacci polynomials modulo 2. We deduce that, when $(n, j) \neq (3, 3)$ and $n \not\equiv 0 \pmod 4$, the Fibonacci knot $\mathcal{F}_j^{(n)}$ is not a Lissajous knot.

1. INTRODUCTION

Fibonacci knots (or links) were defined by J. C. Turner [14] as rational knots with Conway notation $\mathcal{C}(1, 1, \ldots, 1)$. He also considered the generalized Fibonacci knots $\mathcal{F}_j^{(n)} = \mathcal{C}(n, n, \ldots, n)$, where n is a fixed integer and the sequence $(n, \ldots, n)$ has length j.

In this paper we determine the Conway and Alexander polynomials modulo 2 of Fibonacci knots. We show that the Conway polynomial of a generalized Fibonacci knot is a Fibonacci polynomial modulo 2.

As an application, we show that if $(n, j) \neq (3, 3)$ and $n \not\equiv 0 \pmod 4$ the Fibonacci knot $\mathcal{F}_j^{(n)}$ is not a Lissajous knot.

Our results are obtained by continued fraction expansions.

2. CONWAY NOTATION AND FIBONACCI KNOTS

The Conway notation [3] is particularly convenient for the important class of rational (or two-bridge) knots. The Conway normal form $\mathcal{C}(a_1, a_2, \ldots, a_m)$ of a rational knot (or link), is best explained by the following figure. The number of twists is denoted by the integer $|a_i|$,

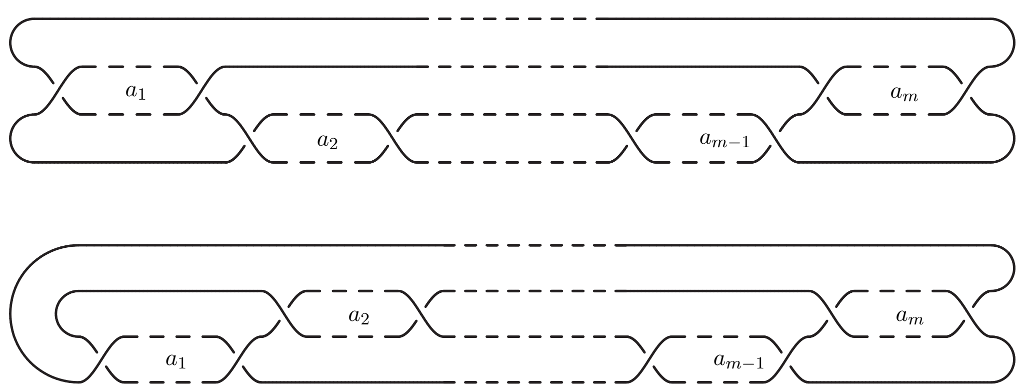

FIGURE 1. Conway's normal forms, m odd (top), m even (bottom).

and the sign of a_i is defined as follows: if i is odd, then the right twist is positive, if i is even, then the right twist is negative. In Figure 1, we see the two cases m odd and m even. In both cases, the a_i are positive (the a_1 first twists are right twists).

Schubert discovered the spectacular classification of rational links ([13], see also [7]). He introduced the Schubert fraction of a rational link:

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_m}}}} = [a_1, \ldots, a_m], \quad \alpha > 0. \tag{2.1}$$

Theorem (Schubert [13])**.** *Two rational links of fractions* $\frac{\alpha}{\beta}$ *and* $\frac{\alpha'}{\beta'}$ *are equivalent if and only if* $\alpha = \alpha'$ *and* $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$.

He also proved that the integer α is the classical determinant of the link, it is odd for a knot, and even for a multi-component link [12, p. 105]. Note that rational links have at most two components.

The following result is a useful consequence of the continued fraction description of rational links [4, p. 207].

Theorem 2.1. *Any rational link has a Conway normal form* $\mathcal{C}(2a_1, 2a_2, \ldots, 2a_m)$.

The Fibonacci knots (or links) are defined by their Conway notation $\mathcal{F}_j = \mathcal{C}(1, 1, \ldots, 1)$, where j is the number of crossings. The Schubert fraction of $\mathcal{F}_j$ is $\frac{F_{j+1}}{F_j}$, and its determinant is the Fibonacci number F_{j+1}. It is the reason why J. C. Turner named these knots Fibonacci knots. He also introduced the generalized Fibonacci knots $\mathcal{F}_j^{(n)} = \mathcal{C}(n, n, \ldots, n)$, where n is a fixed integer.

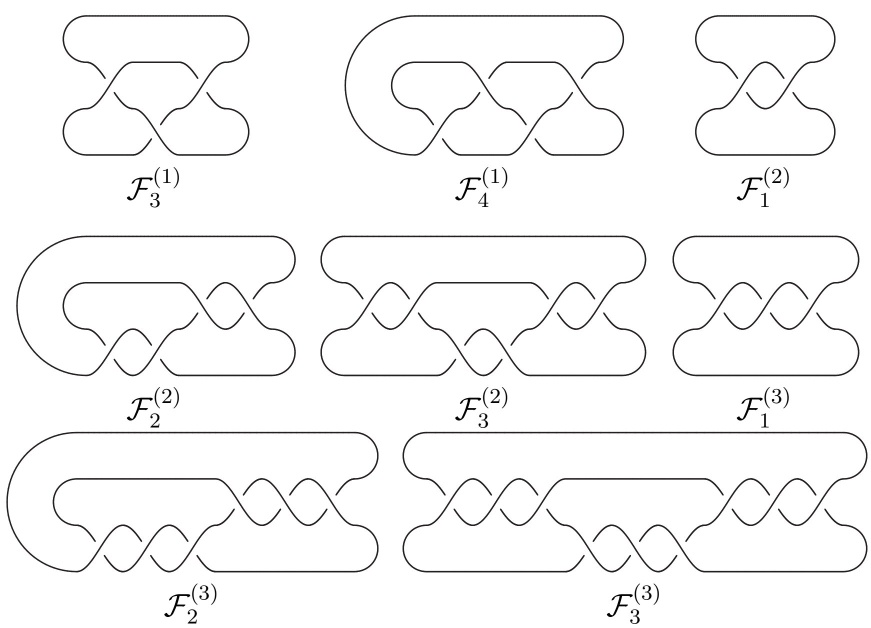

Figure 2. Some Fibonacci knots and links.

We first observe the following proposition.

Proposition 2.2. *$\mathcal{F}_j^{(n)}$ is a knot if and only if $n \equiv 0 \pmod 2$ and $j \equiv 0 \pmod 2$ or $n \not\equiv 0 \pmod 2$ and $j \not\equiv 2 \pmod 3$.*

Proof. Let us consider the Möbius transformation

$$P(z) = [n, z] = n + \frac{1}{z} = \frac{nz+1}{z}. \tag{2.2}$$

Using the standard convention $\dfrac{1}{\infty} = 0$, we get $P(\infty) = n$, $P^2(\infty) = [n, n]$, etc. It is convenient to consider its matrix notation $P = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$.

Let (α, β) be defined by $\dfrac{\alpha}{\beta} = [n, \ldots, n] = P^j(\infty)$, $(\alpha, \beta) = 1$. It is also given by the matrix identity

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P^j \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.3}$$

If $n \equiv 1 \pmod 2$ then $P \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2$, $P^2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \pmod 2$, $P^3 \equiv \mathbb{1} \pmod 2$. We deduce that $\alpha \equiv \beta \equiv 1 \pmod 2$ when $j \equiv 1 \pmod 3$, $\alpha \equiv 0$, $\beta \equiv 1 \pmod 2$ when $j \equiv 2 \pmod 3$ and $\alpha \equiv 1$, $\beta \equiv 0 \pmod 2$ when $j \equiv 0 \pmod 3$. The case $n \equiv 0 \pmod 2$ is similar. $\square$

3. The Conway and Alexander Polynomials

The Alexander polynomial, discovered in 1928, is one of the most famous invariants of knots. J. H. Conway discovered an easy way to calculate it. He introduced the "Skein relations" which relate the polynomial of a link K to the polynomials of links obtained by changing one crossing of K.

The following result is a beautiful application of his algorithm.

Theorem 3.1. [4] *Let $K = \mathcal{C}(2a_1, 2a_2, \ldots, 2a_m)$ be a rational knot (or link). The Conway polynomial of K is*

$$\nabla_K(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -a_1 z & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 z & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} (-1)^m a_m z & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.1}$$

The Alexander polynomial of K is

$$\Delta_K(t) = \nabla_K\bigl(t^{1/2} - t^{-1/2}\bigr). \tag{3.2}$$

Let us consider a simple example.

Example 3.2 (The torus links)**.** *The torus link $\mathrm{T}(2, m)$ has Conway normal form $\mathcal{C}(m) = \mathcal{F}_1^{(m)}$. It is the link of fraction $\dfrac{m}{1}$ or $\dfrac{m}{1-m}$. We have the continued fraction (of length $m-1$)*

$$\frac{m}{1-m} = [-2, 2, \ldots, (-1)^{m-1} \cdot 2]. \tag{3.3}$$

Hence, the Conway polynomial is

$$\nabla(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.4}$$

It is well-known that

$$\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}^m = \begin{pmatrix} f_{m+1}(z) & f_m(z) \\ f_m(z) & f_{m-1}(z) \end{pmatrix} \tag{3.5}$$

where $f_m(z)$ *are the Fibonacci polynomials defined by* $f_0(z) = 0,\ f_1(z) = 1,\ f_{m+1}(z) = zf_m(z) + f_{m-1}(z)$ [15].

We conclude that the Conway polynomial of $\mathrm{T}(2, m)$ *is the Fibonacci polynomial* $f_m(z)$ (see also [6]). If $m = 2k + 1$ (*i.e.* $\mathrm{T}(2, m)$ is a knot) we obtain the Alexander polynomial

$$\Delta(t) = f_{2k+1}\big(t^{1/2} - t^{-1/2}\big) = (t^k + t^{-k}) - (t^{k-1} + t^{k-1}) + \cdots + (-1)^k. \tag{3.6}$$

The recently introduced Lissajous knots [2, 4, 5, 11] are nonsingular Lissajous space curves. We will show that in many cases, Fibonacci knots are not Lissajous knots. Let us first recall the following theorem.

Theorem 3.3. [5, 11] *If* K *is a rational Lissajous knot then* $\Delta_K(t) \equiv 1 \pmod 2$.

Consequently, we see that a nontrivial torus knot is never a Lissajous knot. Moreover, Theorem 3.1 provides many examples of knots which are not Lissajous knots.

Corollary 3.4. *Let* $b_i \equiv 2 \pmod 4, m > 1$. *The Conway polynomial of* $\mathcal{C}(b_1, \ldots, b_m)$ *is equivalent to* $f_{m+1}(z) \pmod 2$.

Corollary 3.5. *If* $n \equiv 2 \pmod 4$, *the modulo* 2 *Conway polynomial of* $\mathcal{F}_j^{(n)}$ *is* $f_{j+1}(z)$.

Hence these knots are not Lissajous knots by Theorem 3.3.

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.6. *If* $n \equiv 0 \pmod 4$, *the modulo* 2 *Conway polynomial of* $\mathcal{F}_j^{(n)}$ *is* 0 *if* j *is odd, and* 1 *if* j *is even.*

It is not known whether the knot $\mathcal{F}_2^{(4)} = \mathcal{C}(4, 4)$ is Lissajous or not (see [1]).

4. The Modulo 2 Conway Polynomial of Fibonacci Knots

We shall now study the knots $\mathcal{F}_j^{(n)}$, where $n = 2k + 1$ is an odd integer.

Lemma 4.1. *Let* $n = 2k + 1$. *We have the identities*

$$[n, n, x] = [n + 1, \underbrace{-2, 2, \ldots, -2, 2}_{2k}, -(1 + x)] \tag{4.1}$$

$$[n, n, n, z] = [n + 1, \underbrace{-2, 2, \ldots, -2, 2}_{2k}, -(n + 1), -z]. \tag{4.2}$$

Proof. Let us prove the first formula. We shall use matrix notations for Möbius transformations. Let $G(u) = [-2, 2, u] = \dfrac{3u + 2}{-2u - 1}$. Its matrix is $G = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, and consequently we get, by induction

$$G^k = \begin{pmatrix} 1 + 2k & 2k \\ -2k & 1 - 2k \end{pmatrix} = \begin{pmatrix} n & n - 1 \\ 1 - n & 2 - n \end{pmatrix}. \tag{4.3}$$

Let

$$M(x) = [n + 1, -2, 2, \ldots, -2, 2, -(1 + x)],\ L(u) = [n + 1, u],\ T(x) = -x - 1. \tag{4.4}$$

The corresponding matrices are

$$L = \begin{pmatrix} n + 1 & 1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},\ M = LG^kT. \tag{4.5}$$

Consequently

$$M = \begin{pmatrix} n+1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n & n-1 \\ 1-n & 2-n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} n^2+1 & n \\ n & 1 \end{pmatrix} = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}^2, \tag{4.6}$$

that is $M(x) = [n, n, x]$ which proves the first identity. If we substitute $x = [n, z]$ in Formula (4.1), we obtain the second identity (4.2). □

Corollary 4.2. *Let $n = 2k + 1$. We have the continued fractions*

$$[n, n] = [n+1, \underbrace{-2, 2, \ldots, -2, 2}_{2k}], \;\; [n, n, n] = [n+1, \underbrace{-2, 2, \ldots, -2, 2}_{2k}, -(n+1)]. \tag{4.7}$$

Let us denote $[n]_j = [\underbrace{n, \ldots, n}_{j}]$. If $j \not\equiv 1 \pmod 3$, we get the continued fractions

$$[n]_{j+3} = [n+1, \underbrace{-2, 2, \ldots, -2, 2}_{2k}, -(n+1), -[n]_j]. \tag{4.8}$$

When $j \equiv 1 \pmod 3$, there is no continued fraction expansion of $[n]_j$ with even quotients, by Proposition 2.2. In this case, we shall get a continued fraction expansion for $\dfrac{\alpha}{\beta - \alpha}$, which is another fraction of the same knot. Let s be the Möbius transformation defined by $s(x) = \dfrac{x}{1-x}$. We have $s(\dfrac{\alpha}{\beta}) = \dfrac{\alpha}{\beta - \alpha}$.

Proposition 4.3. *Let $n = 2k + 1$. We have the continued fractions*

$$s([n]_1) = s(n) = \frac{n}{1-n} = [\underbrace{-2, 2, \ldots, -2, 2}_{2k}], \;\; n \neq 1, \tag{4.9}$$

$$s([n]_{j+3}) = [\underbrace{-2, 2, \ldots, -2, 2}_{2k}, -(n+1), -(n+1), -s([n]_j)]. \tag{4.10}$$

Proof. The first formula has already been proved. Let us prove the second formula. We shall use the Möbius maps $G(x) = [-2, 2, x] = \dfrac{3x+2}{-2x-1}$, $Q(x) = [-(n+1), x]$, $R(x) = -x$ corresponding to

$$G = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \; Q = \begin{pmatrix} -(n+1) & 1 \\ 1 & 0 \end{pmatrix}, \; R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.11}$$

Let us define the Möbius transformation $H = G^k \cdot Q^2 \cdot R$. We obtain, using Formula (4.3)

$$H = \begin{pmatrix} n^3 + n^2 + 2n + 1 & n^2 + 1 \\ -n^3 - n & n - n^2 - 1 \end{pmatrix}. \tag{4.12}$$

Let S be a matrix corresponding to the Möbius map s. We have

$$S^{-1} H S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} H \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} n^3 + 2n & n^2 + 1 \\ n^2 + 1 & n \end{pmatrix} = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}^3, \tag{4.13}$$

and then

$$S \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}^3 = HS. \tag{4.14}$$

This means that $s\big([n, n, n, x]\big) = h \circ s(x)$, which proves our formula. □

Remark 4.4. *By considering the case $n = 1$ where $h(x) = [-2, -2, -x]$, we obtain the following interesting continued fractions of length $2m$:*

$$\frac{F_{3m+2}}{F_{3m}} = [2, 2, -2, -2, \ldots, (-1)^{m-1} \cdot 2, (-1)^{m-1} \cdot 2]. \tag{4.15}$$

Using Corollary 4.2 we obtain similarly

$$\frac{F_{3m+1}}{F_{3m}} = [2, -2, -2, 2, 2, \ldots, (-1)^{m-1} \cdot 2, (-1)^{m-1} \cdot 2, (-1)^{m} \cdot 2] \tag{4.16}$$

of length $2m$ and $\dfrac{F_{3m+3}}{F_{3m+2}} = [2, -\dfrac{F_{3m+2}}{F_{3m}}]$ of length $2m + 1$.

Of course, these fractions correspond to Fibonacci knots (or links). They are not Lissajous knots because of Corollary 3.4.

It is straightforward to calculate the Conway polynomials of our Fibonacci knots, using Proposition 4.3.

Theorem 4.5. *Let us denote by $\nabla_j^{(n)}(z)$ the modulo 2 Conway polynomial of the Fibonacci link $\mathcal{F}_j^{(n)}$. We have $\nabla_j^{(n)}(z) = f_N(z)$ where*

$$\begin{cases} \text{If } n \equiv 1 \pmod 4, & N = \lfloor \frac{j+2}{3} \rfloor (n-2) + j + 1, \\ \text{If } n \equiv 3 \pmod 4, & N = \lfloor \frac{j+2}{3} \rfloor (n+2) - (j+1). \end{cases} \tag{4.17}$$

Corollary 4.6. *If $n \not\equiv 0 \pmod 4$ and $(n, j) \neq (3, 3)$, the Fibonacci link $\mathcal{F}_j^{(n)}$ is not a Lissajous knot.*

It is not known whether the knot $\mathcal{F}_3^{(3)} = \mathcal{C}(3, 3, 3)$ is a Lissajous knot [1].

Question 4.7. *It would be interesting to study the wider classes of knots defined by their Conway notation $\mathcal{C}(\pm n, \pm n, \ldots, \pm n)$.*

If $n = 1$ we obtain all the rational knots [9, 10]. If $n = 2$ we obtain the important class of rational fibered knots [8].

In general, we obtain knots with fractions $\dfrac{\alpha}{\beta}$ such that $(\alpha, \beta) \equiv (0, \pm 1)$ or $(\pm 1, 0) \pmod n$.

5. Acknowledgement

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